CIE A/AS Level Topicals

9231 PAPER 1

FURTHER PURE MATHS

EXAM-READY SERIES

2008 to 2022

TOPICAL PAPERS

- 350 Past-exam Questions
- Solutions
- Sorted by topic
- Examiner's Comments

Visit:
waltermaths.com
for more learning resources.

waltermaths.com

Disclaimer

The copyright and intellectual property owner of the material contained in this booklet is Cambridge Assessment International Education (CAIE). Our role is to curate these questions into organized topics, with the intention of benefiting students.
Walter Maths is not affiliated with and neither is it the official website for any of the stated organization. Therefore,for official announcements/news/updates regarding these qualifications, please visit: https://www.cambridgeinternational.org/

Contents

1. Roots of polynomial equations 3

   Roots of polynomial equations - Solutions 18

2. Rational functions and graphs 41

   Rational functions and graphs - Solutions 57

3. Summation of series 89

   Summation of series - Solutions 104

4. Matrices 126

   Matrices - Solutions 133

5. Polar coordinates 142

   Polar coordinates - Solutions 159

6. Vectors 191

   Vectors - Solutions 209

7. Proof by induction 252

   Proof by induction - Solutions 263

1. Roots of polynomial equations

Q1

5 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots α^3, β^3, γ^3 is

$$y^3 - 3y^2 + 4y - 1 = 0. \quad [4]$$

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

9231/01/M/J/08 Q5

Q2.

The roots of the equation

$$x^4 - 5x^2 + 2x - 1 = 0$$

are α, β, γ, δ. Let $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

(i) Show that

$$S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0. \quad [2]$$

(ii) Find the values of S_2 and S_4. [3]

(iii) Find the value of S_3 and hence find the value of S_6. [6]

(iv) Hence find the value of

$$\alpha^2(\beta^4 + \gamma^4 + \delta^4) + \beta^2(\gamma^4 + \delta^4 + \alpha^4) + \gamma^2(\delta^4 + \alpha^4 + \beta^4) + \delta^2(\alpha^4 + \beta^4 + \gamma^4). \quad [3]$$

9231/01/O/N/08 Q12

Q3.

1 The equation

$$x^4 - x^3 - 1 = 0$$

has roots α, β, γ, δ. By using the substitution $y = x^3$, or by any other method, find the exact value of $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$. [5]

9231/01/M/J/09 Q1

1. Roots of polynomial equations

Q4.

5 The equation

$$x^3 + 5x + 3 = 0$$

has roots α, β, γ. Use the substitution $x = -\frac{3}{y}$ to find a cubic equation in y and show that the roots of this equation are $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$. [4]

Find the exact values of $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$ and $\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3$. [5]

9231/01/O/N/09 Q5

Q5.

6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots α^2, β^2, γ^2. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

9231/11/M/J/10 Q6

Q6.

6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots α^2, β^2, γ^2. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

9231/12/M/J/10 Q6

1. Roots of polynomial equations

Q7.

10 The equation

$$x^4 + x^3 + cx^2 + 4x - 2 = 0,$$

where c is a constant, has roots $\alpha, \beta, \gamma, \delta$.

(i) Use the substitution $y = \frac{1}{x}$ to find an equation which has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$. [2]

(ii) Find, in terms of c, the values of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [3]

(iii) Hence find

$$\left(\alpha - \frac{1}{\alpha}\right)^2 + \left(\beta - \frac{1}{\beta}\right)^2 + \left(\gamma - \frac{1}{\gamma}\right)^2 + \left(\delta - \frac{1}{\delta}\right)^2$$

in terms of c. [2]

(iv) Deduce that when $c = -3$ the roots of the given equation are not all real. [3]

9231/13/M/J/10 Q10

Q8.

7 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha+1}, \frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$. [2]

For the cases $n = 1$ and $n = 2$, find the value of

$$\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}.$$ [2]

Deduce the value of $\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}$. [2]

Hence show that $\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$. [3]

9231/01/O/N/10 Q7

Q9.

2 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are $\frac{\beta}{k}, \beta, k\beta$, where p, q, r, k and β are non-zero real constants. Show that $\beta = -\frac{q}{p}$. [4]

Deduce that $rp^3 = q^3$. [2]

9231/11/M/J/11 Q2

1. Roots of polynomial equations

Q10.

3 Find a cubic equation with roots α, β and γ, given that

$$\alpha + \beta + \gamma = -6, \qquad \alpha^2 + \beta^2 + \gamma^2 = 38, \qquad \alpha\beta\gamma = 30. \qquad [3]$$

Hence find the numerical values of the roots. [3]

9231/13/M/J/11 Q3

Q11.

1 The equation $x^3 + px + q = 0$ has a repeated root. Prove that $4p^3 + 27q^2 = 0$. [5]

9231/11/O/N/11 Q1

Q12.

3 The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

9231/13/O/N/11 Q3

Q13.

1 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$, [2]

(ii) $\alpha^3 + \beta^3 + \gamma^3$. [3]

9231/11/M/J/12 Q1

Q14.

8 The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β, γ.

(i) Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$, $\alpha - \beta + \gamma$, $\alpha + \beta - \gamma$. [5]

(ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$. [5]

9231/13/M/J/12 Q8

1. Roots of polynomial equations

Q15.

The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are α, β, γ, δ, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0. \quad [2]$$

Find the values of

(i) S_2 and S_4, [3]

(ii) S_3 and S_5. [6]

Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3). \quad [3]$$

9231/11/O/N/12 Q11

Q16.

7 A cubic equation has roots α, β and γ such that

$$\alpha + \beta + \gamma = 4,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 34.$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

9231/13/O/N/12 Q7

Q17.

3 The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ. Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]

Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]

Find a cubic equation whose roots are $\frac{1}{\alpha + \beta}, \frac{1}{\beta + \gamma}, \frac{1}{\gamma + \alpha}$. [2]

Hence evaluate $\frac{1}{(\alpha + \beta)^2} + \frac{1}{(\beta + \gamma)^2} + \frac{1}{(\gamma + \alpha)^2}$. [2]

9231/11/M/J/13 Q3

1. Roots of polynomial equations

Q18.

2 The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that α^2, β^2, γ^2 and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0. \quad [3]$$

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72. \quad [3]$$

9231/13/M/J/13 Q2

Q19.

2 The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β, γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$, [1]

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$, [2]

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$. [3]

9231/11/O/N/13 Q2

Q20.

5 The equation

$$8x^3 + 36x^2 + kx - 21 = 0,$$

where k is a constant, has roots $a - d$, a, $a + d$. Find the numerical values of the roots and determine the value of k. [8]

9231/13/O/N/13 Q5

Q21.

1 The equation $x^3 + px + q = 0$, where p and q are constants, with $q \neq 0$, has one root which is the reciprocal of another root. Prove that $p + q^2 = 1$. [5]

9231/11/M/J/14 Q1

1. Roots of polynomial equations

Q22.

The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$, [1]

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, [2]

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$, [2]

(iv) $\frac{\alpha}{\beta\gamma\delta} + \frac{\beta}{\alpha\gamma\delta} + \frac{\gamma}{\alpha\beta\delta} + \frac{\delta}{\alpha\beta\gamma}$. [2]

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$. [7]

9231/11/O/N/14 Q11

Q23.

4 The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of

(i) $\frac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$,

(ii) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$,

(iii) $\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\alpha\beta^2\gamma} + \frac{1}{\alpha\beta\gamma^2}$. [6]

Deduce a cubic equation, with integer coefficients, having roots $\frac{1}{\alpha\beta}$, $\frac{1}{\beta\gamma}$ and $\frac{1}{\gamma\alpha}$. [2]

9231/11/M/J/15 Q4

Q24.

1 The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

9231/13/M/J/15 Q1

Q25.

5 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

9231/11/O/N/15 Q5

1. Roots of polynomial equations

Q26.

1 The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}$, $1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

9231/11/M/J/16 Q1

Q27.

8 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Show that the cubic equation with roots $\frac{\alpha - 1}{\alpha}$, $\frac{\beta - 1}{\beta}$ and $\frac{\gamma - 1}{\gamma}$ may be found using the substitution $z = \frac{1}{1 - x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

9231/13/M/J/16 Q8

Q28.

2 Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

9231/11/O/N/16 Q2

Q29.

7 By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\alpha + \beta + \gamma = -1,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 29,$$
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -1.$$ [8]

9231/11/M/J/17 Q7

1. Roots of polynomial equations

Q30.

1 The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \frac{1}{x^2}$, find the cubic equation with roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

(ii) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. [1]

(iii) Find also the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$. [1]

9231/13/M/J/17 Q1

Q31.

4 The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)$. [4]

(ii) Find the value of $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$. [4]

9231/11/O/N/17 Q4

Q32.

4 It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots a, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

(ii) Deduce the value of k. [2]

9231/11/M/J/18 Q4

1. Roots of polynomial equations

Q33.

6 The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$ [2]

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

(iii) Find the value of S_{-2}. [4]

9231/13/M/J/18 Q6

Q34.

2 The roots of the equation

$$x^3 + px^2 + qx + r = 0$$

are α, 2α, 4α, where p, q, r and α are non-zero real constants.

(i) Show that

$$2p\alpha + q = 0.$$ [4]

(ii) Show that

$$p^3 r - q^3 = 0.$$ [2]

9231/11/O/N/18 Q2

Q35.

1 The roots of the cubic equation

$$x^3 - 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

9231/12/O/N/18 Q1

1. Roots of polynomial equations

Q36.

6 The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots α^3, β^3, γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

9231/11/M/J/19 Q6

Q37.

9 A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

(iv) Given that $3 + i\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

9231/13/M/J/19 Q9

1. Roots of polynomial equations

Q38.

7 The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$. [3]

(iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$. [2]

9231/11/O/N/19 Q7

Q39.

4 The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ.

(a) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

(c) Find a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$, giving your answer in the form

$$px^3 + qx^2 + rx + s = 0,$$

where p, q, r and s are constants to be determined. [3]

9231/01/SP/20 Q4

Q40.

2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

9231/11/M/J/20 Q2

1. Roots of polynomial equations

Q41.

1 The cubic equation $7x^3+3x^2+5x+1=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2}+\beta^{-2}+\gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3}+\beta^{-3}+\gamma^{-3}$. [2]

9231/13/M/J/20 Q1

Q42.

3 The cubic equation $x^3+cx+1=0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

(b) Show that $\alpha^6+\beta^6+\gamma^6=3-2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

9231/11/O/N/20 Q3

Q43.

1 The cubic equation $x^3+bx^2+cx+d=0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma=-1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$. [3]

(c) Given also that $\gamma+1=-\alpha-1$, deduce that $(c-2b+3)(b-3)=b-c$. [4]

9231/12/O/N/20 Q41

Q44.

3 The equation $x^4-2x^3-1=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3+\beta^3+\gamma^3+\delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

9231/11/M/J/21 Q3

1. Roots of polynomial equations

Q45.

2 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$. [2]

9231/13/M/J/21 Q2

Q46.

It is given that

$$\alpha+\beta+\gamma = 3, \quad \alpha^2+\beta^2+\gamma^2 = 5, \quad \alpha^3+\beta^3+\gamma^3 = 6.$$

The cubic equation $x^3 + bx^2 + cx + d = 0$ has roots α, β, γ.

Find the values of b, c and d. [6]

9231/11/O/N/21 Q1

Q47.

The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3 + (\beta+r)^3 + (\gamma+r)^3\right) = n + \tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

9231/12/O/N/21 Q4

1. Roots of polynomial equations

Q48.

The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9} + \frac{1}{\beta^9} + \frac{1}{\gamma^9}$. [2]

9231/11/M/J/22 Q4

Q49.

The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

9231/13/M/J/22 Q2

Q50.

The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

9231/11/O/N/22 Q1

Q51.

The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$. [2]

9231/12/O/N/22 Q2

.

Solutions

Q1.

5	Uses substitution $y = x^3$	M1
	Obtains $y + y^{1/3} - 1 = 0$	A1
	$y = (1 - y)^3$	A1
	$\Rightarrow \ldots \Rightarrow y^3 - 3y^2 + 4y - 1 = 0$ (AG)	A1
	$\sum\alpha^6 = \left(\sum\alpha^3\right)^2 - 2\sum\beta^3\gamma^3$	B1
	$= 9 - 8 = 1$	M1A1
	OR put $y = z^{1/2}$ to obtain	
	$z^3 - z^2 + 10z - 1 = 0$	M1A1
	$\sum\alpha^6 =$ –coefficient of z^2, $= 1$	A1

Q2.

(i)	α a root of given equation $\Rightarrow \alpha^4 - 5\alpha^2 + 2\alpha - 1 = 0$	
	$\Rightarrow \alpha^{n+4} - 5\alpha^{n+2} + 2\alpha^{n+1} - \alpha^n = 0$	M1
	Summing over α, β, γ, δ, leads to $S_{n+4} - 5S_{n+2} + 2S_{n+1} - S_n = 0$	A1
(ii)	$S_2 = 10$	B1
	$S_4 = 5S_2 - 2S_1 + 4 = 50 - 0 + 4 = 54$	M1A1
(iii)	$S_{-1} = 2$ from e.g., $y^4 - 2y^3 + 5y^2 - 1 = 0$	M1A1
	$S_3 = 5S_1 - 2S_0 + S_{-1} = -6$	M1A1
	OR	
	$2S_3 = 3S_1S_2 - S_1^3 + 6\sum\alpha\beta\gamma$	M1A1
	$= 3 \times 10 \times 0 - 0 + 6 \times (-2)$	M1
	$\Rightarrow S_3 = -6$	A1
	$S_6 = 5S_4 - 2S_3 + S_2 = 292$	M1A1
(iv)	$\sum\alpha^2\beta^4 = S_2S_4 - S_6 = 540 - 292 = 248$	M2A1

1. Roots of polynomial equations

Q3.

1 $x = y^{1/3} \Rightarrow y^4 = (1+y)^3$ M1A1

$\Rightarrow y^4 - y^3 - 3y^2 - 3y - 1 = 0$ A1

$\sum \alpha^6 = 1 - 2\times(-3) = 7$ M1A1

or $y = x^2 \Rightarrow y^4 - y^3 - 2y^2 + 1 = 0$ has roots $\alpha^2,\ \beta^2,\ \gamma^2$ M1

$\sum \alpha^6 = \left(\sum \alpha^2\right)\left(\sum \alpha^4\right) - \left(\sum \alpha^2\right)\left(\sum \alpha^2\beta^2\right) + 3\left(\sum \alpha^2\beta^2\gamma^2\right)$ M1A2

$= 5 - (-2) + 0 = 7$ A1

or For last 2 marks

$z = y^2 \Rightarrow z^4 - 7z^3 + z^2 - 3z + 1 = 0$ M1

$\sum \alpha^6 = -\dfrac{(-7)}{1} = 7$ A1

or Use of $S_{N+4} = S_N + S_{N+3}$ M1

$S_{-1} = \dfrac{0}{1} = 0$ $\quad S_2 = 1^2 - 2\times 0 = 1$ (both) M1A1

$S_3 = 0 + 1 = 1$
$S_4 = 1 + 4 = 5$ (both) A1

$S_5 = 5 + 1 = 6$
$S_6 = 6 + 1 = 7$ (both) A1

Q4.

5 Shows $y = -3/x \Rightarrow y^3 - 5y^2 - 9 = 0$ B1
$y = -3/x \Rightarrow y = \alpha\beta\gamma/x$ M1
$\Rightarrow y = \beta\gamma,\ \gamma\alpha,\ \alpha\beta$ when $x = \alpha,\ \beta,\ \gamma$, respectively M1A1

OR for previous 3 marks:
$\beta\gamma + \gamma\alpha + \alpha\beta = 5$ B1
$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma) = \alpha\beta\gamma \times 0 = 0$ B1
$\beta\gamma\gamma\alpha\alpha\beta = (\alpha\beta\gamma)^2 = 9$ B1

$\Sigma\alpha^2\beta^2 = 25 - (2 \times 0) = 25$ M1A1
$\Sigma\alpha^3\beta^3 - 5\,\Sigma\alpha^2\beta^2 - 27 = 0$ M1A1
$\Rightarrow \ldots \Rightarrow \Sigma\alpha^3\beta^3 = 152$ A1

1. Roots of polynomial equations

Q5.

6 Obtains an equation in y not involving radicals, e.g.,

$y(y+1)^2 = 1$ M1

$\Rightarrow \ldots \Rightarrow y^3 + 2y^2 + y - 1 = 0$ (AG) A1

[2]

(i) $S_2 = -2$ B1

$S_4 = 4 - 2 = 2$ M1A1

[3]

(ii) $S_6 = -2S_4 - S_2 + 3 = 1$ M1A1

OR

$\Sigma\alpha^2 = -2,\ \Sigma\alpha^2\beta^2 = 1,\ \alpha^2\beta\gamma^2 = 1$

$S_6 = (\Sigma\alpha^2)^3 - 3\Sigma\alpha^2\Sigma\alpha^2\beta^2 + 3\alpha^2\beta^2\gamma^2$ M1

$= (-2)^3 - 3 \times (-2) \times 1 + 3$

$= -8 + 6 + 3$

$= 1$ A1

$S_8 = -2S_6 - S_4 + S_2 = -6$ M1A1

[4]

Q6.

6 Obtains an equation in y not involving radicals, e.g.,

$y(y+1)^2 = 1$ M1

$\Rightarrow \ldots \Rightarrow y^3 + 2y^2 + y - 1 = 0$ (AG) A1

[2]

(i) $S_2 = -2$ B1

$S_4 = 4 - 2 = 2$ M1A1

[3]

(ii) $S_6 = -2S_4 - S_2 + 3 = 1$ M1A1

OR

$\Sigma\alpha^2 = -2,\ \Sigma\alpha^2\beta^2 = 1,\ \alpha^2\beta\gamma^2 = 1$

$S_6 = (\Sigma\alpha^2)^3 - 3\Sigma\alpha^2\Sigma\alpha^2\beta^2 + 3\alpha^2\beta^2\gamma^2$ M1

$= (-2)^3 - 3 \times (-2) \times 1 + 3$

$= -8 + 6 + 3$

$= 1$ A1

$S_8 = -2S_6 - S_4 + S_2 = -6$ M1A1

[4]

1. Roots of polynomial equations

Q7.

10 **(i)** $x = 1/y \Rightarrow 2y^4 - 4y^3 - cy^2 - y - 1 = 0$ M1A1

[2]

(ii) $\sum\alpha^2 = 1 - 2c$ M1A1

$\sum\alpha^{-2} = 4 + c$ A1

(M1 is for use of $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ in either part.) [3]

(iii) $S = \sum\left(\alpha - \alpha^{-1}\right)^2 = \sum\alpha^2 + \sum\alpha^{-2} - 8 = -c - 3$ M1A1√

A1ft is for adding answers to **(ii)** correctly and subtracting 8. [2]

(iv) $c = -3 \Rightarrow S = 0$ so that if all roots are real then $\alpha = \pm 1$

and similarly for β, γ, δ M1A1 CWO

This is impossible since e.g., $\alpha\beta\gamma\delta = -2$, or any other contradiction A1 CWO

[3]

Q8.

7 $y = \dfrac{1}{x+1} \quad \therefore x = \dfrac{1-y}{y}$ M1 use in given cubic equation

Gives $6y^3 - 7y^2 + 3y - 1 = 0$ **AG** A1 [2]

$n = 1$: given expression = sum of roots = 7/6 B1

$n = 2: \sum\dfrac{1}{(\alpha+1)^2} = \left(\sum\dfrac{1}{(\alpha+1)}\right)^2 - 2\sum"\alpha\beta" = \dfrac{13}{36}$ B1 [2]

From cubic in y,

$6\sum\left(\dfrac{1}{\alpha+1}\right)^3 - 7.\dfrac{13}{36} + 3\left(\dfrac{7}{6}\right) - 3 = 0$ M1

$\sum\left(\dfrac{1}{\alpha+1}\right)^3 = 73/216$ A1 [2]

$\text{LHS} = \sum\left(\dfrac{(\beta+1)(\gamma+1)(\alpha+1}{(\alpha+1)^3}\right)$ M1

$= \left(\dfrac{1}{6}\right)^{-1} \times \dfrac{73}{216}$ M1 recognise product of roots

$= 73/36$ **AG** A1 [3]

1. Roots of polynomial equations

Q9.

2	Sum of roots.	$\frac{\beta+\beta k+\beta k^2}{k}=-p$	B1		
	Sum of products in pairs.	$\frac{\beta^2}{k}+k\beta^2+\beta^2=q$	B1		
	Factorises.	$\Rightarrow \beta\left(\frac{k^2+k+1}{k}\right)=-p$ and $\beta^2\left(\frac{k^2+k+1}{k}\right)=q$	M1		
		$\Rightarrow \beta=-\frac{q}{p}$ (AG)	A1	4	
	Product of roots.	$\beta^3=-r$	B1		
		$\Rightarrow -\frac{q^3}{p^3}=-r \Rightarrow rp^3=q^3$ (AG)	B1	2	**[6]**

Q10.

3	Uses $\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$ States equation with required roots.	$36=38+2\sum\alpha\beta \Rightarrow \sum\alpha\beta=-1$ $\therefore t^3+6t^2-t-30=0$ is the required equation.	M1A1 A1	3	
	Factorises Gives values of α, β, γ.	$\Rightarrow (t-2)(t+3)(t+5)=0$ Hence α, β, and γ are 2, –3 and –5 (in any order). N.B. Answers written down with no working get B1.	M1A1 A1	3	**[6]**

Q11.

1	(N.B. Not α, β, γ)	Let roots be α, α, and β.			
	Writes down sum of roots,	(1) $2\alpha+\beta=0$	M1		
	sum of products in pairs	(2) $2\alpha\beta+\alpha^2=p$	A1		
	and product of roots.	(3) $\alpha^2\beta=-q$	A1		
		From (1) $\beta=-2\alpha$			
	Eliminates β (or α).	(2) $\Rightarrow -4\alpha^2+\alpha^2=p \Rightarrow p=-3\alpha^2$			
		(3) $\Rightarrow -2\alpha^3=-q \Rightarrow q=2\alpha^3$	M1		
	Equates power of α (or β)	$\alpha^6=\left(-\frac{p}{3}\right)^3=\left(\frac{q}{2}\right)^2 \Rightarrow 4p^3+27q^2=0$ (AG)	A1	5	**[5]**

1. Roots of polynomial equations

Q12.

3	Uses	$\sum\alpha=-5 \quad \sum\alpha\beta=-3$	B1		
	$\sum\alpha^2=(\sum\alpha)^2-2\sum\alpha\beta$	$\sum\alpha^2=(-5)^2-2\times(-3)=31$	M1A1	3	
	Evaluates determinant.	$\text{Det}\begin{pmatrix}1 & \alpha & \beta\\ \alpha & 1 & \gamma\\ \beta & \gamma & 1\end{pmatrix}=1-(\alpha^2+\beta^2+\gamma^2)+2\alpha\beta\gamma$	M1A1		
		$\alpha\beta\gamma=-(-15)=15$			
		$\Rightarrow 1-31+2\times 15$	M1		
	Shows it is zero.	$=0 \Rightarrow$ matrix is singular.	A1	4	[7]

Q13.

1	States $\sum\alpha$ and $\sum\alpha\beta$	$\sum\alpha=7 \quad \sum\alpha\beta=2$	B1		
	Uses formula for correctly.	$\sum\alpha^2=7^2-2\times 2=45$	B1	2	
	Uses formula for $\sum\alpha^3$	$\sum\alpha^3=7\sum\alpha^2-2\sum\alpha+9$	M1		
	to obtain result.	$=315\text{-}14\text{+}9=310$	A1A1	3	[5]

1. Roots of polynomial equations

Q14.

	Alternatively:				
8 (i)	**For final 3 marks in (i):** Award M1 for an attempt at formulae for all three coefficients. A1 for any two correct. A1 for completion	Let equation be $u^3 + bu^2 + cu + d = 0$. $-b = \sum\alpha = 1 \Rightarrow b = -1$ $c = 4\sum\alpha\beta - (\sum\alpha)^2$ $= 4 \times (-3) - 1^2 = -13$ $-d = 4\sum\alpha\sum\alpha\beta - \left(\sum\alpha\right)^3 - 8\alpha\beta\gamma$ $= 4 \times 1 \times (-3) - 1^3 - 8 \times 10 = -93$ So $u^3 - u^2 - 13u + 93 = 0$		(5)	
(ii)	**For final 4 marks in (ii):** Award M1 for an attempt at formulae for all three coefficients. A1 for any one correct. A1 for a second one correct. A1 for completion.	Let equation be $v^3 + bv^2 + cv + d = 0$. $-b = \frac{\Sigma\alpha}{\alpha\beta\gamma} = \frac{1}{10} \Rightarrow b = -\frac{1}{10}$ $c = \frac{\Sigma\alpha\beta}{(\alpha\beta\gamma)^2} = \frac{-3}{10^2} = -\frac{3}{100}$ $-d = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{10^2} \Rightarrow d = -\frac{1}{100}$ So $v^3 - \frac{1}{10}v^2 - \frac{3}{100}v - \frac{1}{100} = 0$ or $100v^3 - 10v^2 - 3v - 1 = 0$.		(5)	**[10]**

Q15.

11	**EITHER** Substitute α into equation. Multiply by α^n. Obtain result.	α is a root $\Rightarrow \alpha^4 - 3\alpha^2 + 5\alpha - 2 = 0$ $\Rightarrow \alpha^{n+4} - 3\alpha^{n+2} + 5\alpha^{n+1} - 2\alpha^n = 0$ Repeat for β, γ, δ and sum $\Rightarrow S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$ (AG)	M1 A1	2	
(i)	Uses $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$ Finds S_4 from formula.	$S_2 = 0 - 2 \times (-3) = 6$ $S_4 = 3 \times 6 - 5 \times 0 + 2 \times 4 = 26$	B1 M1A1	3	
(ii)	$S_{-1} = \frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}$ Finds S_3 from formula. Finds S_5 from formula.	$S_{-1} = \frac{-5}{-2} = \frac{5}{2}$ $S_3 = 3 \times 0 - 5 \times 4 + 2 \times \frac{5}{2} = -15$ $S_5 = 3 \times (-15) - 5 \times 6 + 2 \times 0 = -75$ $\sum\alpha^2\beta^3 = S_2S_3 - S_5$ $= 6 \times (-15) - (-75) = -15$	M1A1 M1A1 M1A1 M1 M1A1	6 3	**[14]**

1. Roots of polynomial equations

Q16.

7		$(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=\alpha^2+\beta^2+\gamma^2 \Rightarrow \sum\alpha\beta=1$	M1A1	2	
		Either			
		Required equation is $x^3-4x^2+x+c=0$	M1		
		$\Rightarrow \sum\alpha^3-4\sum a^2+4+3c=0$	M1		
		$\Rightarrow 3c=56-34-4=18 \Rightarrow c=6$ (AG)	A1		
		Or			
		$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$	(M1)		
		(or some other appropriate identity, e.g.			
		$(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma)$	(M1A1)		
		$\Rightarrow \ldots \Rightarrow \alpha\beta\gamma=-6$			
		$\Rightarrow x^3-4x^2+x+6=0$ (AG)			
		$\Rightarrow (x+1)(x-2)(x-3)=0 \Rightarrow x=-1,2,3.$	A1		
			M1A1	6	**[8]**

Q17.

3	Uses $\sum\alpha=\frac{-b}{a}$.	$c=2$	B1	1	
	Uses substitution	$(\alpha+\beta=c-\gamma$ etc.) $\Rightarrow y=c-x \Rightarrow x=c-y$	M1		
		$(2-y)^3-2(2-y)^2-3(2-y)+4=0$ …(their c)	M1		
	to obtain required cubic equation.	$\Rightarrow y^3-4y^2+y+2=0$	A1	3	
	Obtains equation whose roots are reciprocals of those in previous cubic equation.	Uses $z=y^{-1}$ to obtain $2z^3+z^2-4z+1=0$	M1A1	2	
	Uses $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$	$\sum\frac{1}{(\alpha+\beta)^2}=\left(\frac{1}{2}\right)^2-2(-2)=4\frac{1}{4}$	M1A1	2	**[8]**

1. Roots of polynomial equations

Q18.

2	Makes substitution. Squares. Obtains result.	$y^2 - 4y + 3y^{\frac{1}{2}} - 2 = 0$ $\Rightarrow 9y = 4 + y^4 + 16y^2 - 4y^2 + 16y - 8y^3$ (N.B. Must see both terms in y^2.) $\Rightarrow y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$ (AG)	M1 M1 A1	 3	
		$S_2 = 0^2 - 2\times(-4) = 8$ $S_8 = 8S_6 - 12S_4 - 7S_2 - 16$ $\Rightarrow S_8 = 8S_6 - 12S_4 - 56 - 16 = 8S_6 - 12S_4 - 72$ (AG) **Alternatively** – for final two marks. $S_2 = 8$, $S_3 = -9$, $S_4 = 40$, $S_5 = -60$, $S_6 = 203$, $S_7 = -378$ $S_8 = 1072$ (generated by substitution of roots in equations and summing.) Then $8S_6 - 12S_4 - 72 = 1624 - 480 - 72 = 1072 = S_8$ M1 requires a complete method, A1 if all correct.	B1 M1 A1	 3	**[6]**

Q19.

2 (i)	Finds S_2.	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ $= 0 - 2(-p) = 2p$ (AG)	B1	1	
(ii)	Finds S_3.	$\alpha^3 + \beta^3 + \gamma^3 = p\sum\alpha + 3q = 0 + 3q = 3q$ (AG)	M1A1	2	
(iii)	Finds S_5.	$\alpha^5 + \beta^5 + \gamma^5 = p\sum\alpha^3 + q\sum\alpha^2$ $= p.3q + q.2p = 5pq$ $\Rightarrow 6\sum\alpha^5 = 30\text{pq} = 5\sum\alpha^3\sum\alpha^2$ (AG)	M1 A1 A1	 3	**[6]**

1. Roots of polynomial equations

Q20.

5	Uses sum of roots	$\sum\alpha = 3a = -\frac{36}{8} = -\frac{9}{2} \Rightarrow a = -\frac{3}{2}$	M1A1		
	Uses product of roots	$\alpha\beta\gamma = a\left(a^2 - d^2\right) = \frac{21}{8}$	M1		
	Substitutes for a,	$\frac{9}{4} - d^2 = -\frac{2}{3} \times \frac{21}{8} = -\frac{7}{4}$	M1		
	and solves	$\Rightarrow d^2 = 4 \Rightarrow d = \pm 2$	A1		
		Roots are $-\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}$	A1		
	Uses sum of products in pairs. (Or expands $(2x+7)(2x+3)(2x+1)$ and equates coefficients)	$\sum\alpha\beta = \frac{21}{4} - \frac{7}{4} - \frac{3}{4} = \frac{k}{8}$	M1		
		$\Rightarrow k = 22$	A1	**(8)**	
					[8]

Q21.

1	Let roots be α, α^{-1} and β	$\Rightarrow \alpha + \alpha^{-1} + \beta = 0$	(1)	Any of three for M1	M1
	Product of roots	$\Rightarrow \beta = -q$	(2)	A1 for another.	A1
	Sum of products in pairs	$\Rightarrow 1 + \beta(\alpha + \alpha^{-1}) = p$	(3)	A1 for a third	A1
	From (1) and (3)	$1 - \beta^2 = p$		Wrong sign in (2) scores	M1
	Using (2)	$1 - q^2 = p$ or $p + q^2 = 1$		M1A0A1M1A0	A1
					[5]

1. Roots of polynomial equations

Q22.

11E (i)	$\alpha+\beta+\gamma+\delta=-4$	B1 (1)
(ii)	$\alpha^2+\beta^2+\gamma^2+\delta^2=(-4)^2-2\times2=12$	M1A1 (2)
(iii)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{-(-4)}{1}=4$	M1A1 (2)
(iv)	$\frac{\alpha}{\beta\gamma\delta}+\frac{\beta}{\alpha\gamma\delta}+\frac{\gamma}{\alpha\beta\delta}+\frac{\delta}{\alpha\beta\gamma}=\frac{\alpha^2+\beta^2+\gamma^2+\delta^2}{\alpha\beta\gamma\delta}=\frac{12}{1}=12$	M1A1 (2)
	$y=x+1\Rightarrow x=y-1$	
	$(y-1)^4+4(y-1)^3=y^4-6y^2+8y-3$	M1A1
	$2(y-1)^2-4(y-1)+1=2y^2-8y+7$	A1
	$\Rightarrow x^4+4x^3+2x^2-4x+1=y^4-4y^2+4=0$	A1
	$(y^2-2)^2=0\Rightarrow y=\pm\sqrt{2}$ (twice).	A1
	$\Rightarrow x=\pm\sqrt{2}-1$ (twice). (Some indication of four roots for final mark.)	M1A1 (7) **[14]**

Q23.

4	$\alpha+\beta+\gamma=7$, $\alpha\beta+\beta\gamma+\gamma\alpha=2$, $\alpha\beta\gamma=3$ (Stated or implied by working.)	B1
(i)	$\frac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}=\frac{1}{(\alpha\beta\gamma)^2}=\frac{1}{9}$	B1
(ii)	$\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{7}{3}$	M1A1
(iii)	$\frac{1}{\alpha^2\beta\gamma}+\frac{1}{\alpha\beta^2\gamma}+\frac{1}{\alpha\beta\gamma^2}=\frac{1}{\alpha\beta\gamma}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)=\frac{1}{\alpha\beta\gamma}\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right)=\frac{2}{9}$	M1A1 (6)
	$\Rightarrow x^3-\frac{7}{3}x^2+\frac{2}{9}x-\frac{1}{9}=0\Rightarrow 9x^3-21x^2+2x-1=0$	M1A1√ (2) **Total: 8**

1. Roots of polynomial equations

Q24.

1	i $2\alpha+2\beta=0$ ii $\alpha^2+4\alpha\beta+\beta^2=-p$ iii $2\alpha^2\beta+2\alpha\beta^2=-q$ iv $\alpha^2\beta^2=-r$ All four correct (Any two correct) Use of $\beta=-\alpha$ in ii, iii or iv $\Rightarrow p=2\alpha^2$ and $\alpha^4=-r\Rightarrow p^2+4r=0$ (AG) and $q=0$ (CAO)	B2 (B1) M1 M1A1 B1 (6) **Total 6**

Q25.

5	$\alpha+\beta+\gamma=-p=15\Rightarrow p=-15$	**B1**
	$2(\alpha\beta+\beta\gamma+\gamma\alpha)=(\alpha+\beta+\gamma)^2-\left(\alpha^2+\beta^2+\gamma^2\right)=2q$	**M1**
	$\Rightarrow q=\frac{1}{2}(225-83)=71$	**A1** [3]
	$\frac{36}{\alpha}=15-\alpha \quad (=[\beta+\gamma])$	**M1**
	$\Rightarrow a^2-15\alpha+36=0\Rightarrow\alpha=3,\ \alpha\neq12$, ***e.g.*** since $12^2>83$ or other reason	**M1A1**
	$\beta\gamma=71-36=35$	**B1**
	$\Rightarrow r=-\alpha\beta\gamma=-3\times35=-105$ (extra answer penalised)	**A1** [5] **Total 8**

Q26.

1	$y=1+\frac{1}{x}\Rightarrow x=\frac{1}{y-1}$	**M1**
	$\frac{2}{(y-1)^3}+\frac{1}{(y-1)^2}-7=0\Rightarrow 2+(y-1)-7(y-1)^3=0$	**A1**
	$\Rightarrow 7\left(y^3-3y^2+3y-1\right)-y+1-2=0\Rightarrow 7y^3-21y^2+20y-8=0$	**M1A1** [4]
	ALT METHOD: ∑α, ∑αβ, αβγM1 A1, ∑(1+1/α) etc M1 A1	

1. Roots of polynomial equations

Q27.

8	Let S_n denote $\alpha^n + \beta^n + \gamma^n$. $S_2 = S_1^2 - 2\sum\alpha\beta = 1^2 - 2\times(-1) = 3$ $S_3 = S_2 + S_1 + 15 = 3 + 1 + 15 = 19$ **(AG)** Alt method: $\sum\alpha = 1$, $\Sigma\alpha\beta = -1$, and $\alpha\beta\gamma = 5$ **M1, A1** $S_3 = S_1^3 - 3\Sigma\alpha\,\Sigma\alpha\beta + 3\alpha\beta\gamma = 19$ **M1, A1** $S_4 = S_3 + S_2 + 5S_1 = 19 + 3 + 5 = 27$ $x = \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow \frac{1}{z} = 1 - x \Rightarrow z = \frac{1}{1-x}$ Hence $\frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{1}{(1-x)} - 5 = 0$ is the required equation. $\Rightarrow 1 - (1-x) - (1-x)^2 - 5(1-x)^3 = 0 \Rightarrow \ldots \Rightarrow 5x^3 - 16x^2 + 18x - 6 = 0$	**M1A1** **M1A1** [4] **M1A1** [2] **B1** **M1** **M1A1** [4]

Q28.

2	$2\sum\alpha\beta = 9 - 1 \Rightarrow \sum\alpha\beta = 4$	**M1A1**	
	Use of, e.g.: $\sum\alpha^3 - 3\alpha\beta\gamma = \sum\alpha\left(\sum\alpha^2 - \sum\alpha\beta\right)$	**M1**	
	or $\left(\sum\alpha\right)^3 = \sum\alpha^3 + 3\sum\alpha\sum\alpha\beta - 3\alpha\beta\gamma$		
	Correct substitution in formula	**A1**	
	$\Rightarrow \alpha\beta\gamma = -7$	**A1**	
	Required cubic equation is $x^3 - 3x^2 + 4x + 7 = 0$ *must see final equation*	**A1**√	[6]
	ALT METHOD: $S_3 - 3S_2 + 4S_1 + 3r = 0$ **M1** $3r = 30 + 3\times 1 - 4\times 3$ **A1** $r = 7$ **A1**		

Q29.

7	$2\sum\alpha\beta = 1 - 29 \Rightarrow \sum\alpha\beta = -14$	**M1A1**	
	$\frac{\sum\alpha\beta}{\alpha\beta\gamma} = \frac{-14}{\alpha\beta\gamma} = -1 \Rightarrow \alpha\beta\gamma = 14$	**M1A1 FT**	
	$\Rightarrow x^3 + x^2 - 14x - 14 = 0$	**A1**	
	$\Rightarrow (x+1)(x^2 - 14)$	**M1A1**	Attempt to factorise cubic
	⇒Solution is -1, in $\pm\sqrt{14}$ any order. Accept ± 3.74 (awrt) SR **B1** for correct roots without working	**A1**	
	Total:	**8**	

1. Roots of polynomial equations

Q30.

1(i)	$y=\frac{1}{x^2}\Rightarrow x=\frac{1}{\sqrt{y}}$	**M1**	Rearranging to make x the subject
	$\frac{1}{y\sqrt{y}}+\frac{2}{y}-3=0\Rightarrow\frac{1}{y\sqrt{y}}=3-\frac{2}{y}$	**M1**	Substituting and squaring
	$\Rightarrow 9y^3-12y^2+4y-1=0$ SR **B1** for finding cubic by manipulating roots	**A1**	OE
	Total:	**3**	
1(ii)	$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{12}{9}$ or $\frac{4}{3}$	**B1FT**	
	Total:	**1**	
1(iii)	$\frac{1}{\alpha^2\beta^2}+\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2}=\frac{4}{9}$	**B1FT**	
	Total:	**1**	

Q31.

4(i)	$\alpha+\beta+\gamma=\frac{3}{2}\quad\alpha\beta+\beta\gamma+\gamma\alpha=2\quad\alpha\beta\gamma=5+\beta+\gamma=$ $\frac{3}{2}\alpha\beta+\beta\gamma+\gamma\alpha=2\alpha\beta\gamma=5$	**B1**	(Can be awarded in (ii) if not seen here) SOI
	$(\alpha+1)(\beta+1)(\gamma+1)=\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)+$ $(\alpha+\beta+\gamma)+1$	**M1A1**	Multiply out **and** group for **M1**
	$=5+2+1\frac{1}{2}+1=9\frac{1}{2}$	**A1FT**	Alt method: Let $x=y-1$ **M1** Sub and expand $2y^3-9y^2\ 16y-19=0$ **M1, A1** Product of roots = 19/2 **A1**
		4	
4(ii)	$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)=\left(1\frac{1}{2}-\alpha\right)\left(1\frac{1}{2}-\beta\right)\left(1\frac{1}{2}-\gamma\right)$	**M1**	Alt methods: $=(\Sigma\alpha)(\Sigma\alpha\beta)-\alpha\beta\gamma$ or $\Sigma\alpha^2\Sigma\alpha+2\alpha\beta\gamma-\Sigma\alpha^3$
	$=\frac{27}{8}-\frac{9}{4}(\alpha+\beta+\gamma)+\frac{3}{2}(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$	**A1**	
	$=\frac{27}{8}-\frac{9}{4}\times\frac{3}{2}+\frac{3}{2}\times 2-5=-2$	**M1A1**	
		4	

1. Roots of polynomial equations

Q32.

4(i)	$\alpha\beta\gamma = a^3 = 216 \Rightarrow a = 6$	**M1 A1**	Uses product of roots
	$a + ar + ar^{-1} = 21$ $6\left(1 + r + r^{-1}\right) = 21$	**M1**	Uses sum of roots
	$2r^2 - 5r + 2 = 0 \Rightarrow r = 2$ or $r = 0.5$	**M1 A1**	Substitutes for a and solves quadratic
	Roots are 6, 12, 3	**A1**	
		6	
4(ii)	$k = \alpha\beta + \alpha\gamma + \beta\gamma = 6(12) + 6(3) + 12(3) = 126$	**M1 A1**	Or finds coefficient of x in $(x-3)(x-6)(x-12)$. Or substitutes root into equation
		2	

Q33.

6(i)	Substitutes $x = \frac{y+1}{3}$	**M1**	Accept substitution of $y = 3x - 1$ into given equation and derivation of equation in x.
	Obtains the given result	**A1**	AG.
6(ii)	$S_3 = 2S_1 + 7 \times 3$	**M1**	Uses $y^3 = 2y + 7$. Or uses formula for $\Sigma\,(3\alpha - 1)^3$
	$= 21$	**A1**	
6(iii)	$S_{-1} = \frac{(3\alpha-1)(3\beta-1)+(3\alpha-1)(3\gamma-1)+(3\beta-1)(3\gamma-1)}{(3\alpha-1)(3\beta-1)(3\gamma-1)} = \frac{-2}{7}$.	**M1 A1**	Award M1A1 if $S_{-1} = -\frac{2}{7}$ written down directly.
	$7S_{-2} = S_1 - 2S_{-1}$	**M1**	Uses $7y^{-2} = y - 2y^{-1}$.
	$s_{-2} = \frac{4}{49}$	**A1**	
	Alt method: $S_{-2} = \Sigma\frac{1}{(3\alpha-1)^2} = \frac{\Sigma(3\alpha-1)^2(3\beta-1)^2}{(3\alpha-1)^2(3\beta-1)^2(3\gamma-1)^2} =$	**M1 A1**	**Alt method**: Finds cubic with roots $\frac{1}{3\alpha-1}$, etc. M1 $7z^3 + 2z^2 - 1 = 0$ A1 Uses $S_2 = (S_1)^2 - 2x\Sigma\alpha\beta$ M1 $= \frac{4}{49}$ A1
	$\frac{(\Sigma(3\alpha-1)(3\beta-1))^2 - 2(3\alpha-1)(3\beta-1)(3\gamma-1)(\Sigma(3\alpha-1))}{(3\alpha-1)^2(3\beta-1)^2(3\gamma-1)^2}$	**M1**	
	$= \frac{(-2)2 - 2(7)(0)}{7^2} = \frac{4}{49}$	**A1**	
		8	

1. Roots of polynomial equations

Q34.

2(i)	$\alpha+2\alpha+4\alpha=-p$	**B1**	Sum of roots.
	$2\alpha^2+4\alpha^2+8\alpha^2=q$	**B1**	Sum of products in pairs.
	$\frac{14\alpha^2}{7\alpha}=-\frac{q}{p}$	**M1**	Combines equations.
	$\Rightarrow 2p\alpha+q=0$	**A1**	Verifies result (AG).
		4	
2(ii)	$8\alpha^3=-r$	**B1**	Product of roots.
	$\Rightarrow r=\frac{q^3}{p^3} \Rightarrow p^3r-q^3=0$	**B1**	Verifies result (AG).
		2	

Q35.

1(i)	$\alpha+\beta+\gamma=5,\ \alpha\beta+\alpha\gamma+\beta\gamma=13$	**B1**	Sum of roots and $\alpha\beta+\alpha\gamma+\beta\gamma$. SOI
	$\alpha^2+\beta^2+\gamma^2=5^2-2(13)$	**M1**	Uses $\Sigma\alpha^2=(\Sigma\alpha)^2-2(\Sigma\alpha\beta)$
	$=-1$	**A1**	www
		3	
1(ii)	$\alpha^3+\beta^3+\gamma^3=5\left(\alpha^2+\beta^2+\gamma^2\right)-13(\alpha+\beta+\gamma)+12$	**M1**	Uses $\alpha^3=5\alpha^2-13\alpha+4$.
	$=5(-1)-13(5)+12=-58$	**A1**	
	Alt method: Use formula e.g. $\Sigma\alpha^3=(\Sigma\alpha)\left(\Sigma\alpha^2-\Sigma\alpha\beta\right)+3\alpha\beta\gamma$ Or $(\Sigma\alpha)^3-3(\Sigma\alpha)(\Sigma\alpha\beta)+3\alpha\beta\gamma$		
		2	

Q36.

6(i)	$y=(y+1)^3$	**M1**	Obtains an equation in y not involving radicals.
	$y=y^3+3y^2+3y+1\Rightarrow y^3+3y^2+2y+1=0$	**A1**	AG
	$S_3=-3$	**B1**	
		3	
6(ii)	$S_{-3}=\frac{\alpha^3\beta^3+\beta^3\gamma^3+\alpha^3\gamma^3}{\alpha^3\beta^3\gamma^3}=\frac{2}{-1}=-2$	**M1 A1**	
		2	
6(iii)	$S_6=(-3)^2-2(2)=5$	**M1 A1**	Uses $(\Sigma\alpha)^2=\Sigma\alpha^2+2\sum_{\alpha\neq\beta}\alpha\beta$. AG.
	$S_9=-3S_6-2S_3-3=-3(5)-2(-3)-3=-12$	**M1 A1**	Sums $y^3-3y^2+2y-1=0$ for $y=\alpha^3,\beta^3,\gamma^3$.
		4	

1. Roots of polynomial equations

Q37.

9(i)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}\Rightarrow c=\alpha\beta+\beta\gamma+\alpha\gamma=5$	**M1 A1**	Uses $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$.
	$d=12$	**B1**	
		3	
9(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1**	
	$(\alpha+\beta+\gamma)^2-10=b^2-10$	**A1**	
	Alternative method for 9(ii)		
	$90+bS_2-5b+36=0\Rightarrow S_2=\frac{5b-126}{b}$	**M1 A1**	Sums over roots
		2	

9(iii)	$x^3+bx^2+5x+12=0$ $\quad(-b=\alpha+\beta+\gamma)$	**M1**	Formulates equation.
	$90+b\left(b^2-10\right)-5b+36=0$	**M1 A1**	Sums over roots and uses **9(ii)**
	$b^3-15b+126=0$	**A1**	
	Alternative method 1 for 9(iii)		
	$(\alpha+\beta+\gamma)^3=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$	**M1 A1**	Uses expansion of $(\alpha+\beta+\gamma)^3$
	$-b^3=90-15b+36$	**(M1)**	Substitutes known values.
	$b^3-15b+126=0$	**A1**	
	Alternative method 2 for 9(iii)		
	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=b^2-10$	**M1 A1**	Finds sum of squares in terms of b.
	$\frac{5b-126}{b}=b^2-10\Rightarrow b^3-15b+126=0$	**M1 A1**	Equates expressions.
		4	
9(iv)	Real root is $b=-6$	**M1 A1**	Gives the real root for b.

1. Roots of polynomial equations

Q38.

7(i)	$\sqrt{-7y}(-7y)+2(-7y)+\sqrt{-7y}+7=0$ $\Rightarrow\sqrt{-7y}(-7y+1)=14y-7\Rightarrow -7y(-7y+1)^2=(14y-7)^2$	**M1**	Uses given substitution and eliminates radical.
	$\Rightarrow 49y^3+14y^2-27y+7=0$	**A1**	AG
	$y=\frac{x^2}{-7}=\frac{x^2}{\alpha\beta\gamma}$	**M1**	Uses $\alpha\beta\gamma=-7$.
	So roots are $\frac{\alpha^2}{\alpha\beta\gamma}=\frac{\alpha}{\beta\gamma},\frac{\beta^2}{\alpha\beta\gamma}=\frac{\beta}{\alpha\gamma},\frac{\gamma^2}{\alpha\beta\gamma}=\frac{\gamma}{\alpha\beta}$	**A1**	AG
		4	
7(ii)	$\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}=-\frac{2}{7},\frac{1}{\gamma^2}+\frac{1}{\beta^2}+\frac{1}{\alpha^2}=-\frac{27}{49}$	**B1**	States sum of roots and $\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'$.
	$\frac{\alpha^2}{\beta^2\gamma^2}+\frac{\beta^2}{\alpha^2\gamma^2}+\frac{\gamma^2}{\alpha^2\beta^2}=\left(-\frac{2}{7}\right)^2-2\left(-\frac{27}{49}\right)=\frac{58}{49}$	**M1 A1**	Uses $\alpha'^2+\beta'^2+\gamma'^2=(\alpha'+\beta'+\gamma')^2-2(\alpha'\beta'+\alpha'\gamma'+\beta'\gamma')$ AG
		3	
7(iii)	$49\left(\frac{\alpha^3}{\beta^3\gamma^3}+\frac{\beta^3}{\alpha^3\gamma^3}+\frac{\gamma^3}{\alpha^3\beta^3}\right)=-14\left(\frac{58}{49}\right)+27\left(-\frac{2}{7}\right)-21$	**M1**	Uses $49\alpha'^3=-14\alpha'^2+27\alpha'-7$.
	$\Rightarrow\frac{\alpha^3}{\beta^3\gamma^3}+\frac{\beta^3}{\alpha^3\gamma^3}+\frac{\gamma^3}{\alpha^3\beta^3}=-\frac{317}{343}$	**A1**	
		2	

Q39.

Question	Answer	Marks	Partial Marks	Guidance
4(a)	**EITHER** Solution 1 Let S_n denote $\alpha^n+\beta^n+\gamma^n$ $S_2=S_1^2-2\sum\alpha\beta=1^2-2\times(-1)=3$	2	**(M1A1**	
	$S_3=S_2+S_1+15=3+1+15=19$	2	**M1A1)**	
	OR Solution 2 Use $\Sigma\alpha^3=(\Sigma\alpha)^3-3(\Sigma\alpha)(\Sigma\alpha\beta)+3\alpha\beta\gamma$	1	**(M1**	
	$\Sigma\alpha=1,\ \Sigma\alpha\beta=-1,\ \alpha\beta\gamma=5$	1	**B1**	
	$\Sigma\alpha^3=1-3\times1\times-1+3\times5$	1	**M1**	
	$=19$	1	**A1)**	
	Available marks	**4**		
4(b)	$S_4=S_3+S_2+5S_1=19+3+5=27$	**2**	**M1A1**	
4(c)	$x=z+1\Rightarrow z=x-1$	1	**B1**	
	Required eqn: $(x-1)^3-(x-1)^2-(x-1)-5=0$	1	**M1**	
	So, $x^3-4x^2+4x-6=0$	1	**A1**	
		3		

1. Roots of polynomial equations

Q40.

2(a)	$y = x^2$	**B1**
	$6y^{\frac{1}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py + 5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	**M1**
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	**A1**
		3
2(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2+36}{36}$	**B1**
	$\frac{p^2+36}{36} = -\frac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	**M1**
	$p = -6$	**A1**
		3
2(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	**M1**
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	**A1**
		2

Q41.

1(a)	$y = x^{-1}$	**B1**
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	**M1 A1**
		3
1(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	**M1 A1**
1(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	**M1 A1**
		4

Q42.

3(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	**M1**	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	**A1**	
		3	
3(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	**M1**	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	**A1**	AG
		3	

3(c)	$\alpha^3\beta^3\gamma^3=-1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1-\left(\alpha^6+\beta^6+\gamma^6\right)+2\alpha^3\beta^3\gamma^3 = 2c^3-4$	**M1 A1**	Evaluates determinant.
	$2c^3-4=0$	**M1**	Sets determinant equal to zero.
	$c=\sqrt[3]{2}$	**A1**	
		5	

Q43.

1(a)	$d=1$	**B1**	
		1	
1(b)	$y=x+1\Rightarrow x=y-1$	**B1**	Uses correct substitution.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1)=c-2b+3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1)=3-b,\ \ (\alpha+1)(\beta+1)(\gamma+1)=c-b$	**M1**	Using these relationships.
	$y^3+(b-3)y^2+(c-2b+3)y+b-c=0)$	**A1**	
		3	
1(c)	$\beta+1=-(b-3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1)=c-2b+3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1)=-(b-c)$	**M1**	Applies product of roots.
	$\Rightarrow(c-2b+3)(b-3)=b-c$	**A1**	AG
		4	

Q44.

3(a)	$y=x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}}-2y-1=0\Rightarrow y^4=(2y+1)^3=8y^3+12y^2+6y+1$	**M1**	Obtains an equation not involving radicals.
	$y^4-8y^3-12y^2-6y-1=0$	**A1**	
	$\alpha^3+\beta^3+\gamma^3+\delta^3=8$	**B1 FT**	
		4	

3(b)	$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}=\frac{\alpha^3\beta^3\delta^3+\alpha^3\beta^3\gamma^3+\beta^3\gamma^3\delta^3+\alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3}=\frac{6}{-1}$	**M1** **A1 FT**	Relates $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
3(c)	$\alpha^4+\beta^4+\gamma^4+\delta^4=2(\alpha^3+\beta^3+\gamma^3+\delta^3)+4$	**M1**	Uses original equation.
	$=20$	**A1**	
		2	

Q45.

2(a)	$S_1=2$	**B1**	
	$S_2=S_1^2-2(0)$	**M1**	Uses formula for sum of squares.
	$=4$	**A1**	Correct answer implies M1A1.
		3	
2(b)(i)	$S_{n+3}=2S_{n+2}-\frac{3}{2}S_n$	**B1**	CAO or as a single fraction.
		1	
2(b)(ii)	$S_4=2S_3-\frac{3}{2}S_1=2(2S_2-\frac{3}{2}S_0)-\frac{3}{2}S_1$	**M1**	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3=\frac{7}{2}\right]$.
	$=4$	**A1**	
		2	
2(c)	$x=2-y$	**B1**	SOI
	$2(2-y)^3-4(2-y)^2+3=0$	**M1**	Makes *their* substitution.
	$2y^3-8y^2+8y-3=0$	**A1**	OE but must be an equation.
		3	

Q46.

1	$b=-(\alpha+\beta+\gamma)=-3$	**B1**	
	$5=3^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1 A1**	Uses formula for sum of squares.
	$c=2$	**A1**	
	$6-3(5)+2(3)+3d=0$	**M1**	Uses original equation or formula for sum of cubes.
	[Equation is $x^3-3x^2+2x+1=0$] $d=1$	**A1**	
		6	

1. Roots of polynomial equations

Q47.

4(a)	$(-2)^2-2(3)$	**M1**	Uses formula for sum of squares.
	-2	**A1**	
		2	
4(b)	$\alpha^3+\beta^3+\gamma^3=-2(-2)-3(-2)-3(3)$	**M1**	Uses original equation or formula for sum of cubes.
	1	**A1**	AG
		2	

4(c)	$(\alpha+r)^3=\alpha^3+3\alpha^2r+3\alpha r^2+r^3$	**B1**	Expands.
	$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=\sum_{r=1}^{n}\left(1+3(-2)r+3(-2)r^2+3r^3\right)$	**M1 A1**	Collects like terms and uses results from parts **(a)** and **(b)**.
	$n-6\left(\frac{1}{2}n(n+1)\right)-6\left(\frac{1}{6}n(n+1)(2n+1)\right)+\frac{3}{4}n^2(n+1)^2$ $n-3n(n+1)-n(n+1)(2n+1)+\frac{3}{4}n^2(n+1)^2$	**M1**	Applies formulae from MF19.
	$n+\frac{1}{4}n(n+1)(-12-4(2n+1)+3n(n+1))$ $n+\frac{1}{4}n(n+1)\left(3n^2-5n-16\right)$	**M1 A1**	Simplifies.
		6	

Q48.

4(a)	$y=x^{-3}\Rightarrow x=y^{-\frac{1}{3}}$	**B1**	Substitutes.
	$\Rightarrow 2y^{-1}+5y^{-\frac{2}{3}}-6=0$ leading to $5y^{-\frac{2}{3}}=6-2y^{-1}\Rightarrow 125y=(6y-2)^3$	**M1**	Cubes to eliminate radical.
	$216y^3-216y^2-53y-8=0$	**A1**	
	Alternative method for question 4(a)		
	$(2x^3-6)^3=(-5x^2)^3$	**M1**	$8x^9-72x^6+125x^6+216x^3-216=0$
	$y=x^{-3}$ leading to $x^3=y^{-1}$	**B1**	Substitutes.
	$216y^3-216y^2-53y-8=0$	**A1**	
		3	
4(b)	$\alpha^{-3}+\beta^{-3}+\gamma^{-3}=1\quad\alpha^{-3}\beta^{-3}+\beta^{-3}\gamma^{-3}+\gamma^{-3}\alpha^{-3}=-\frac{53}{216}$	**B1 FT**	Using *their* answer to part **(a)**.
	$\alpha^{-6}+\beta^{-6}+\gamma^{-6}=1^2-2\left(-\frac{53}{216}\right)$	**M1**	$\alpha^{-6}+\beta^{-6}+\gamma^{-6}=(\alpha^{-3}+\beta^{-3}+\gamma^{-3})^2-2(\alpha^{-3}\beta^{-3}+\beta^{-3}\gamma^{-3}+\gamma^{-3}\alpha^{-3})$
	$\alpha^{-6}+\beta^{-6}+\gamma^{-6}=\frac{161}{108}$	**A1**	
		3	
4(c)	$216S_{-9}=216S_{-6}+53S_{-3}+24$	**M1**	Using *their* $216\alpha^{-9}-216\alpha^{-6}-53\alpha^{-3}-8=0$
	$S_{-9}=\frac{399}{216}=\frac{133}{72}$	**A1**	
		2	

1. Roots of polynomial equations

Q49.

2(a)	$\alpha\beta+\beta\gamma+\gamma\alpha=10$	**B1**	SOI
	$\alpha^2+\beta^2+\gamma^2=5^2-20=5$	**M1 A1**	Uses $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$.
	Alternative method for question 2(a)		
	Put $y=x^2$ and form cubic not involving x	**M1 A1**	$y(y+10)^2=25y^2-20y+4$ (need not be simplified). $y^3-5y^2+120y-4=0$
	$\alpha^2+\beta^2+\gamma^2=[-(-5)]=5$	**A1**	
		3	
2(b)	$\begin{vmatrix}1 & \alpha & \beta\\ \alpha & 1 & \gamma\\ \beta & \gamma & 1\end{vmatrix}=\begin{vmatrix}1 & \gamma\\ \gamma & 1\end{vmatrix}-\alpha\begin{vmatrix}\alpha & \gamma\\ \beta & 1\end{vmatrix}+\beta\begin{vmatrix}\alpha & 1\\ \beta & \gamma\end{vmatrix}=1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma$	**M1 A1**	Finds determinant.
	$1-\alpha^2-\beta^2-\gamma^2+2\alpha\beta\gamma=1-5+2(2)=0$	**M1 A1**	Substitutes and uses $\alpha\beta\gamma=2$
		4	

Q50.

1(a)	$2\alpha+\gamma=-b$ $\alpha^2+2\alpha\gamma=0$	**B1**	
	$\alpha=-2\gamma$ leading to $-4\gamma+\gamma=-b$	**M1**	Solves simultaneous equations, or, express b and d in terms of α and γ.
	$\gamma=\frac{1}{3}b,\ \alpha=-\frac{2}{3}b$	**A1**	$b=3\gamma=-\frac{3}{2}\alpha$ and $d=-\alpha^2\gamma$.
	$\alpha^2\gamma=-d$ leading to $\frac{4}{27}b^3=-d$ leading to $4b^3+27d=0$	**M1 A1**	Substitutes into third equation, AG.
		5	
1(b)	$3b=b^2$ leading to $b=3$	**M1 A1**	Uses $2\alpha^2+\gamma^2=(2\alpha+\gamma)^2-2(\alpha^2+2\alpha\gamma)$ or substitutes for α,γ in terms of b.
	$d=-4$	**A1**	
		3	

1. Roots of polynomial equations

Q51.

2(a)	$y = x^{-2}$ leading to $x = y^{-\frac{1}{2}}$	**B1**	Correct substitution.
	$y^{-2} + 3y^{-1} + 2y^{-\frac{1}{2}} + 6 = 0$ leading to $1 + 3y + 2y^{\frac{3}{2}} + 6y^2 = 0$	**M1**	Obtains an equation not involving radicals.
	$36y^4 + 32y^3 + 21y^2 + 6y + 1 = 0$	**A1**	
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = -\frac{32}{36} = -\frac{8}{9}$	**B1 FT**	
		4	
2(b)	$\alpha\beta\gamma\delta = 6$	**B1**	SOI
	$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2}{\alpha^2\beta^2\gamma^2\delta^2}$	**M1**	Relates to coefficients.
	$\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2\alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2 = -32$	**A1**	
		3	
2(c)	$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} = \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2 - 2\left(\alpha^{-2}\beta^{-2} + \alpha^{-2}\gamma^{-2} + a^{-2}\delta^{-2} + \beta^{-2}\gamma^{-2} + \beta^{-2}\delta^{-2} + \gamma^{-2}\delta^{-2}\right) = \left(-\frac{8}{9}\right)^2 - 2\left(\frac{21}{36}\right)$	**M1**	Uses formula for sum of squares.
	$-\frac{61}{162}$	**A1**	
		2	

2. Rational functions and graphs

Q1.

The curve C has equation

$$y = \frac{x^2 - 2x + \lambda}{x + 1},$$

where λ is a constant. Show that the equations of the asymptotes of C are independent of λ. [3]

Find the value of λ for which the x-axis is a tangent to C, and sketch C in this case. [4]

Sketch C in the case $\lambda = -4$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

9231/01/M/J/08 Q9

Q2.

The curve C has equation

$$y = \frac{(x-2)(x-a)}{(x-1)(x-3)},$$

where a is a constant not equal to 1, 2 or 3.

(i) Write down the equations of the asymptotes of C. [2]

(ii) Show that C meets the asymptote parallel to the x-axis at the point where $x = \frac{2a-3}{a-2}$. [2]

(iii) Show that the x-coordinates of any stationary points on C satisfy

$$(a-2)x^2 + (6-4a)x + (5a-6) = 0,$$

and hence find the set of values of a for which C has stationary points. [6]

(iv) Sketch the graph of C for

(a) $a > 3$,

(b) $2 < a < 3$.

[4]

9231/01/O/N/08 Q12

Q3.

10 The curve C has equation

$$y = \frac{x^2}{x + \lambda},$$

where λ is a non-zero constant. Obtain the equation of each of the asymptotes of C. [3]

In separate diagrams, sketch C for the cases $\lambda > 0$ and $\lambda < 0$. In both cases the coordinates of the turning points must be indicated. [8]

9231/01/M/J/09 Q10

2. Rational functions and graphs

Q4.

3 The curve C has equation

$$y = \frac{x^2 - 5x + 4}{x + 1}.$$

(i) Obtain the coordinates of the points of intersection of C with the axes. [2]

(ii) Obtain the equation of each of the asymptotes of C. [3]

(iii) Draw a sketch of C. [3]

9231/01/O/N/09 Q3

Q5.

The curve C has equation

$$y = \frac{x(x + 1)}{(x - 1)^2}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that there is exactly one point of intersection of C with the asymptotes and find its coordinates. [2]

(iii) Find $\frac{dy}{dx}$ and hence

(a) find the coordinates of any stationary points of C,

(b) state the set of values of x for which the gradient of C is negative.

[6]

(iv) Draw a sketch of C. [3]

9231/11/M/J/10 Q11

Q6.

6 The curve C has equation

$$y = \frac{x^2 - 3x - 7}{x + 1}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that $\frac{dy}{dx} > 1$ at all points of C. [2]

(iii) Draw a sketch of C. [3]

9231/13/M/J/10 Q6

2. Rational functions and graphs

Q7.

The curve C has equation

$$y = \frac{x^2 + 2\lambda x}{x^2 - 2x + \lambda},$$

where λ is a constant and $\lambda \neq -1$.

(i) Show that C has at most two stationary points. [3]

(ii) Show that if C has **exactly** two stationary points then $\lambda > -\frac{5}{4}$. [2]

(iii) Find the set of values of λ such that C has two vertical asymptotes. [2]

(iv) Find the x-coordinates of the points of intersection of C with

(a) the x-axis,

(b) the horizontal asymptote.

[3]

(v) Sketch C in each of the cases

(a) $\lambda < -2$,

(b) $\lambda > 2$.

[4]

9231/01/O/N/10 Q12

Q8.

The curve C has equation

$$y = \frac{x^2 + \lambda x - 6\lambda^2}{x + 3},$$

where λ is a constant such that $\lambda \neq 1$ and $\lambda \neq -\frac{3}{2}$.

(i) Find $\frac{dy}{dx}$ and deduce that if C has two stationary points then $-\frac{3}{2} < \lambda < 1$. [5]

(ii) Find the equations of the asymptotes of C. [3]

(iii) Draw a sketch of C for the case $0 < \lambda < 1$. [3]

(iv) Draw a sketch of C for the case $\lambda > 3$. [3]

9231/11/M/J/11 Q11

2. Rational functions and graphs

Q9

9 The curve C with equation

$$y = \frac{ax^2 + bx + c}{x - 1},$$

where a, b and c are constants, has two asymptotes. It is given that $y = 2x - 5$ is one of these asymptotes.

(i) State the equation of the other asymptote. [1]

(ii) Find the value of a and show that $b = -7$. [3]

(iii) Given also that C has a turning point when $x = 2$, find the value of c. [3]

(iv) Find the set of values of k for which the line $y = k$ does not intersect C. [4]

9231/13/M/J/11 Q9

Q10.

The curve C has equation $y = \frac{x^2 + px + 1}{x - 2}$, where p is a constant. Given that C has two asymptotes, find the equation of each asymptote. [3]

Find the set of values of p for which C has two distinct turning points. [5]

Sketch C in the case $p = -1$. Your sketch should indicate the coordinates of any intersections with the axes, but need not show the coordinates of any turning points. [3]

9231/11/O/N/11 Q7

Q11.

10 A curve C has equation

$$y = \frac{5(x^2 - x - 2)}{x^2 + 5x + 10}.$$

Find the coordinates of the points of intersection of C with the axes. [2]

Show that, for all real values of x, $-1 \leqslant y \leqslant 15$. [4]

Sketch C, stating the coordinates of any turning points and the equation of the horizontal asymptote. [7]

9231/13/O/N/11 Q10

2. Rational functions and graphs

Q12.

9 The curve C has equation

$$y = \frac{2x^2 + 2x + 3}{x^2 + 2}.$$

Show that, for all x, $1 \leqslant y \leqslant \frac{5}{2}$. [4]

Find the coordinates of the turning points on C. [3]

Find the equation of the asymptote of C. [2]

Sketch the graph of C, stating the coordinates of any intersections with the y-axis and the asymptote. [2]

9231/11/M/J/12 Q9

Q13.

6 The curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C. [3]

Find the coordinates of the turning points on C. [3]

Draw a sketch of C. [3]

9231/13/M/J/12 Q6

Q14.

7 The curve C has equation

$$y = \lambda x + \frac{x}{x-2},$$

where λ is a non-zero constant. Find the equations of the asymptotes of C. [3]

Show that C has no turning points if $\lambda < 0$. [3]

Sketch C in the case $\lambda = -1$, stating the coordinates of the intersections with the axes. [3]

9231/11/O/N/12 Q7

Q15.

9 The curve C has equation $y = \frac{x^2 - 3x + 3}{x-2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $-1 < y < 3$. [4]

Find the coordinates of the turning points of C. [3]

Sketch C. [2]

9231/13/O/N/12 Q9

2. Rational functions and graphs

Q16.

10 The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$. State the equations of the asymptotes of C. [2]

Show that $y \leqslant \frac{25}{12}$ at all points of C. [4]

Find the coordinates of any stationary points of C. [3]

Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]

9231/11/M/J/13 Q10

Q17.

10 The curve C has equation

$$y = \frac{px^2 + 4x + 1}{x + 1},$$

where p is a positive constant and $p \neq 3$.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of p for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) For the case $p = 1$, show that C has no turning points, and sketch C, giving the exact coordinates of the points of intersection of C with the x-axis. [5]

9231/11/O/N/13 Q10

Q18.

7 The curve C has equation

$$y = \frac{2x^2 + 5x - 1}{x + 2}.$$

Find the equations of the asymptotes of C. [3]

Show that $\dfrac{dy}{dx} > 2$ at all points on C. [3]

Sketch C. [3]

09231/13/O/N/13 Q7

2. Rational functions and graphs

Q19.

The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + d},$$

where a, b, c and d are constants. The curve cuts the y-axis at $(0, -2)$ and has asymptotes $x = 2$ and $y = x + 1$.

(i) Write down the value of d. [1]

(ii) Determine the values of a, b and c. [6]

(iii) Show that, at all points on C, either $y \leqslant 3 - 2\sqrt{6}$ or $y \geqslant 3 + 2\sqrt{6}$. [7]

9231/11/M/J/14 Q12

Q20.

Express $\frac{2x^2 - x + 5}{x^2 - 1}$ in the form $2 + \frac{A}{x - 1} + \frac{B}{x + 1}$, where A and B are integers to be found. [3]

The curve C has equation $y = \frac{2x^2 - x + 5}{x^2 - 1}$. Show that there are two distinct values of x for which $\frac{dy}{dx} = 0$. [4]

Sketch C, stating the equations of the asymptotes and giving the coordinates of any points of intersection with the coordinate axes and with the asymptotes. You do not need to find the coordinates of the turning points. [7]

9231/13/M/J/14 Q11

Q21.

4 A curve C has equation $y = \frac{2x^2 + x - 1}{x - 1}$. Find the equations of the asymptotes of C. [3]

Show that there is no point on C for which $1 < y < 9$. [4]

9231/11/O/N/14 Q4

2. Rational functions and graphs

Q22.

10 The curve C has equation $y = \dfrac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form $y = -\dfrac{1}{2} + \dfrac{(3x-1)^2}{2(x^2+1)}$ and also in the form $y = \dfrac{9}{2} - \dfrac{(x+3)^2}{2(x^2+1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant y \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C. [2]

State the equation of the asymptote of C. [1]

Sketch the graph of C, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

9231/13/M/J/15 Q10

Q23.

8 The curve C has equation $y = \dfrac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

9231/11/O/N/15 Q8

Q24.

A curve C has equation $y = \dfrac{x^2}{x - 2}$. Find the equations of the asymptotes of C. [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C, giving the coordinates of the turning points. [3]

9231/11/M/J/16 Q7

Q25.

5 The curve C has equation $y = \dfrac{x + 2}{x^2 - 9}$. Show that $\dfrac{dy}{dx} < 0$ at all points on C. [3]

State the equations of the asymptotes of C. [2]

Sketch C, showing the coordinates of any points of intersection with the coordinate axes. [3]

9231/13/M/J/16 Q5

2. Rational functions and graphs

Q26.

9 The curve C has equation $y = \dfrac{x^2 - 3x + 6}{1 - x}$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the turning points of C. [3]

(iii) Find the coordinates of any intersections with the coordinate axes. [2]

(iv) Sketch C. [3]

9231/11/M/J/17 Q9

Q27.

9 The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

(i) Find the equations of the asymptotes of C. [2]

(ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]

(iii) Find the coordinates of the turning points of C. [3]

(iv) Sketch C. [3]

9231/11/O/N/17 Q9

Q28.

6 The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C does not intersect the x-axis. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C. Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points. [3]

9231/11/M/J/18 Q6

2. Rational functions and graphs

Q29.

4 The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

(i) Find the coordinates of the points of intersection of C with the axes. [2]

(ii) Find the equation of each of the asymptotes of C. [3]

(iii) Sketch C. [3]

9231/13/M/J/18 Q4

Q30.

6 The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

(i) Find the equations of the asymptotes of C. [3]

(ii) Show that C intersects the x-axis twice. [1]

(iii) Justifying your answer, find the number of stationary points on C. [2]

(iv) Sketch C, stating the coordinates of its point of intersection with the y-axis. [3]

9231/11/O/N/18 Q6

Q31.

9 The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

(i) Find the equation of the asymptote of C. [2]

(ii) Show that, for all real values of x, $-\frac{1}{3} \leqslant y < 5$. [4]

(iii) Find the coordinates of any stationary points of C. [2]

(iv) Sketch C, stating the coordinates of any intersections with the y-axis. [2]

9231/12/O/N/18 Q9

2. Rational functions and graphs

Q32.

10 The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

(i) Find the equations of the asymptotes of C_1. [2]

(ii) Find the equation of the oblique asymptote of C_2. [2]

(iii) Show that C_1 and C_2 do not intersect. [2]

(iv) Find the coordinates of the stationary points of C_2. [3]

(v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

9231/11/M/J/19 Q10

Q33.

6 The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the coordinates of the stationary points of C. [3]

(iii) Sketch C. [3]

9231/13/M/J/19 Q6

Q34.

4 The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

(i) Find the values of the constants a and b. [3]

(ii) State the equation of the other asymptote of C. [1]

(iii) Sketch C. [Your sketch should indicate the coordinates of any points of intersection with the y-axis. You do not need to find the coordinates of any stationary points.] [3]

9231/11/O/N/19 Q4

2. Rational functions and graphs

Q35.

1 Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$. [4]

9231/11/M/J/20 Q1

Q36.

3 The curve C has equation $y = \frac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch C. [3]

9231/11/M/J/20 Q3

Q37.

6 The curve C has equation $y = \frac{10+x-2x^2}{2x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\frac{10+x-2x^2}{2x-3}\right| < 4$. [6]

9231/13/M/J/20 Q6

2. Rational functions and graphs

Q38.

6 The curve C has equation $y = \dfrac{x^2+x-1}{x-1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x-1}{x-1}\right|$. [2]

9231/11/O/N/20 Q6

Q39.

6 Let a be a positive constant.

(a) The curve C_1 has equation $y = \dfrac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\dfrac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\dfrac{x-a}{x-2a}\right|$.

(b) **(i)** Find the coordinates of any stationary points of C_2. [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\dfrac{x-a}{x-2a}\right)^2 \leqslant \left|\dfrac{x-a}{x-2a}\right|$. [5]

9231/12/O/N/20 Q6

Q40.

7 The curve C has equation $y = \dfrac{x^2+x+9}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2|x^2+x+9| > 13|x+1|$. [5]

9231/11/M/J/21 Q7

2. Rational functions and graphs

Q41.

7 The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

9231/13/M/J/21 Q7

Q42.

The curve C has equation $y = \dfrac{4x + 5}{4 - 4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{4x + 5}{4 - 4x^2}\right|$ and find in exact form the set of values of x for which $4|4x + 5| > 5|4 - 4x^2|$. [6]

9231/11/O/N/21 Q7

Q43.

The curve C has equation $y = \dfrac{x^2}{x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $0 < y < 12$. [4]

(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\dfrac{x^2}{x - 3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\dfrac{x^2}{x - 3}\right| \leqslant |x| + c$ has no solution. [1]

9231/12/O/N/21 Q6

2. Rational functions and graphs

Q44.

The curve C has equation $y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$ has 4 distinct real solutions. [2]

9231/11/M/J/22 Q5

Q45.

(a) Sketch the curve with equation $y = \dfrac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \dfrac{|x|+1}{|x|-1}$ and find the set of values of x for which $\dfrac{|x|+1}{|x|-1} < -2$. [4]

9231/13/M/J/22 Q1

Q46.

A curve C has equation $y = \dfrac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]

(c) Sketch C. You do not need to find the coordinates of the intersections with the axes. [3]

9231/13/M/J/22 Q3

2. Rational functions and graphs

Q47.

The curve C has equation $y = \dfrac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{5x^2}{5x-2}\right| < 2$. [6]

9231/11/O/N/22 Q7

Q48.

The curve C has equation $y = \dfrac{x^2-x}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the exact coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{x^2-x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2-x}{x+1}\right| < 6$. [5]

9231/12/O/N/22 Q7

Solutions

Q1.

9	One asymptote is $x = -1$ (Allow $x \to -1$, $y \to x - 3$)	B1
	$y = x - 3 + (\lambda + 3)/(x + 1)$	M1
	$\Rightarrow y = x - 3$	A1
	$\lambda = 1$	B1
	Axes plus both asymptotes drawn	B1 ft
	DEF: Here Z denotes 'correct shape, orientation and approximately correct location'	
	RH branch with Z	B1 ft
	LH branch with Z	B1
	$\lambda = -4$: RH branch with Z	B1
	LH branch with Z	B1
	Intersections with x-axis $(1 + \sqrt{5}, 0)$, $(1 - \sqrt{5}, 0)$	B1

Q2.

12	**EITHER**	
(i)	$x = 1, x = 3$ (both)	B1
	$y = 1$	B1
(ii)	Solves $(x-2)(x-a)/(x-1)(x-3) = 1$ to obtain $x = \xi$ where $\xi = (2a-3)/(a-2)$	M1A1
(iii)	$y_1 = 0 \Rightarrow (x-2)(x-a)(2x-4) = (x-1)(x-3)(2x-2-a)$	M1
	$\Rightarrow (-4-4-2a)x^2 + (4a+8+4a)x - 8a = (-8-2-a)x^2 + (6+8+4a)x - 6 - 3a$	A1
	$\Rightarrow (a-2)x^2 + (6-4a)x + (5a-6) = 0$ (AG)	A1
	$(6-4a)^2 \geqslant 4(a-2)(5a-6)$	M1
	$\Rightarrow a^2 - 4a + 3 \leqslant 0 \Rightarrow (a-1)(a-3) \leqslant 0$	M1
	$\Rightarrow 1 < a < 3$ ($a \neq 2$ given)	A1
(iv)	**(a)** Axes and asymptotes	B1
	Branches (all)	B1
	(b) Middle branch with maximum value in the range $0 < y < 1$	B1
	Outside branches with correctly placed minimum point	B1

2. Rational functions and graphs

Q3.

10	One asymptote is $x = -\lambda$	B1
	$y = x - \lambda + \lambda^2/(x + \lambda)$	M1
	Other asymptote is $y = x - \lambda$	A1
	or for previous 2 marks:	
	$mx + c = x^2/(x + \lambda) \Rightarrow (m-1)x^2 + (c + \lambda m)x + \lambda c = 0$	
	$m - 1 = 0$, $c + \lambda m = 0$ (both)	M1
	$m = 1$, $c = -\lambda$ (both)	A1
	Differentiates and $y' = 0$	M1
	Identifies turning point at $(-2\lambda, -4\lambda)$ and $(0, 0)$	A1A1
	(Award D1 if correct turning points stated with no working)	
	First graph:	
	Axes and both asymptotes	B1
	Branches	B1B1
	Deduct at most 1, overall, for bad forms at infinity	
	Second graph:	
	Axes and both asymptotes + 1 branch	B1
	Other branch	B1

Deduct at most 1, overall, for bad forms at infinity

Choose which is 1st graph and which is 2nd to the benefit of candidates, i.e. best = 1st

For differentiating a special case M1A0A0

Graphs of special cases B2 (max) B1 (max)

If stated in text, turning points need not be labelled on graph

2. Rational functions and graphs

Q4.

3	**(i)**	(1,0), (4,0)	B1
		(0,4)	B1
	(ii)	One asymptote is $x = -1$	B1
		$y = x - 6 + 10/(x + 1)$	M1
		Other asymptote: $y = x - 6$	A1
	(iii)	Sketch:	
		Axes and asymptotes	B1
		Upper branch:	
		Correct location and orientation	B1
		Lower branch correctly located and orientated	B1

Q5.

11 OR

(i)		$x = 1$	B1
		$y = 1 + O(1/x)$ as $\|x\| \to \infty$	M1
		Second asymptote is $y = 1$	A1
			[3]
(ii)		$x(x + 1) / (x - 1)^2 = 1 \Rightarrow x = 1/3, y = 1$	M1A1
			[2]
(iii)	**(a)**	$dy / dx = 0 \Rightarrow [(2x + 1)(x - 1)^2 - 2x(x - 1)(x + 1)] / (x - 1)^4 = 0$	M1
		$\Rightarrow x = 1/3, y = -1/8$	M1A1
	(b)	$dy / dx = -(3x + 1) / (x - 1)^3$	M1
		$\{x : x < -1/3\} \cup \{x : x > 1\}$	A1√A1
		ft.	[6]
(iv)		Sketch:	
		Left-hand branch with approximately correct shape and location and passing through the origin and (–1, 0).	B1
		Intersection with $y = 1$ and location of minimum point consistent with results of **(ii)** and **(iii)** (cwo)	B1
		Right-hand branch with approximately correct forms at infinity	B1
			[3]

Q6.

6	**(i)**	One asymptote is $x = -1$	B1
		$y = x - 4 - 3/(x + 1)$	M1
		Require $y = x +$ non-zero constant.	
		Other asymptote is $y = x - 4$	A1
			[3]
		Alternatively for last two marks:	
		OR $x + k \approx (x^2 - 3x - 7)/(x + 1)$ for large $x \Rightarrow x^2 + (k + 1)x + k \approx x^2 - 3x - 7$ for large x	M1
		$\Rightarrow k + 1 = -3 \Rightarrow k = -4 \Rightarrow$ other asymptote is $y = x - 4$	A1

OR $(x^2-3x-7)/(x+1)=mx+c \Rightarrow (m-1)x^2+(m+c+3)x+c+7=0$ M1
Put $m-1=0, m+c+3=0$ to obtain other asymptote is $y=x-4$ A1

(ii) Obtains any correct result for dy/dx M1

$\Rightarrow \ldots \Rightarrow dy/dx=\left[(x+1)^2+3\right]/(x+1)^2=1+3/(x+1)^2$ A1
No comment required. [2]

(iii) Axes and asymptotes correctly placed B1
Right-hand branch B1
Left-hand branch B1

Deduct 1 overall for bad form(s) at infinity [3]

Q7.

12 EITHER

(i) $y'=0 \Rightarrow (x^2-2x+\lambda)(2x+2\lambda)-(x^2+2\lambda x)(2x-2)=0$ M1
$\Rightarrow \ldots \Rightarrow (\lambda+1)x^2-\lambda x-\lambda^2=0$ A1
Hence at most 2 values of x and at most 2 stationary points A1 [3]

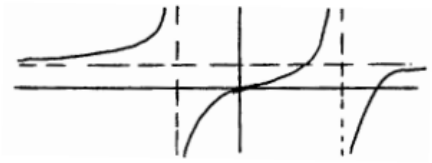
(ii) For 2 real distinct roots, $\lambda^2>4(\lambda+1)(-\lambda^2)$ M1 use of discriminant
$\lambda^2(5+4\lambda)>0 \therefore \lambda>-\frac{5}{4}$ **AG** A1 [2]

(iii) Vert. asymptotes when $x^2-2x+\lambda=0$ M1
$b^2-4ac>0 \Rightarrow 4-4\lambda>0$
For two vert. asymp. $\lambda<1$ A1 [2]

(iv) (a) $y=0 \Rightarrow x^2+2\lambda x=0$ M1
$\Rightarrow x=0$ or -2λ A1 (both)

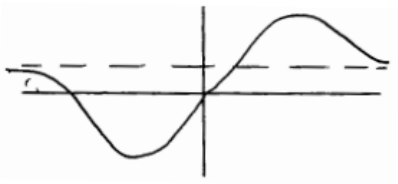
(b) $y=1: x=\frac{\lambda}{2\lambda+2}$ B1 [3]

(v) (a) $\lambda<-2$: no stat points: 2 vert. asymp



B1 3 branches
B1 completely correct shape

(b) $\lambda<2$: 2 stats points: no vert. asymp



B1 max, min, horiz asymp
B1 correct shape [4]

2. Rational functions and graphs

Q8.

11	**OR**				
(i)	Differentiates y wrt x.	$\frac{dy}{dx} = \frac{(x+3)(2x+\lambda)-(x^2+\lambda x-6\lambda^2)}{(x+3)^2}$	M1		
		$= \frac{x^2+6x+3\lambda+6\lambda^2}{(x+3)^2}$	A1		
		$\frac{dy}{dx} = 0$ has distinct roots if	M1		
	Uses discriminant and factorises.	$36-12\lambda-24\lambda^2 > 0 \Rightarrow 3-\lambda-2\lambda^2 > 0$ $(3+2\lambda)(1-\lambda) > 0$	A1		
	Obtains result. (No equality.)	$\Rightarrow -\frac{3}{2} < \lambda < 1$ (AG)	A1	5	
(ii)	Uses division to obtain the form for recognition of oblique asymptote.	$\frac{x^2+\lambda x-6\lambda^2}{x+3} \equiv x+\lambda-3+\frac{9-3\lambda-6\lambda^2}{x+3}$	M1		
		(Tolerate error on remainder term.) $\Rightarrow$ asymptotes are: $x=-3$	B1		
		and $y = x+\lambda-3$.	A1	3	
(iii)	For $0 < \lambda < 1$	Axes and asymptotes.	B1		
		Upper branch with minimum below x-axis.	B1		
		Lower branch with maximum.	B1	3	
(iv)	For $\lambda > 3$	Axes and asymptotes. (n.b. oblique asymptote has positive y intercept.)	B1		
		Left-hand branch.	B1		
		Right-hand branch.	B1	3	
	Deduct 1 mark overall for wrong forms At infinity.				**[14]**

2. Rational functions and graphs

Q9.

9	**(i)**	States vertical asymptote.	$x = 1$	B1	1	
	(ii)	States the value of a.	$a = 2$	B1		
			$y = ax + a + b + \dfrac{a+b+c}{(x-1)}$	M1		
		Divides. Compares coefficients to obtain b.	$2 + b = -5 \Rightarrow b = -7$ (AG)	A1	3	
			Or			
			$y = 2x - 5 + \dfrac{a}{x-1}$	(M1)		
			$= \dfrac{2x^2 - 7x + 5 + a}{x-1}$	(B1A1)		
			Equate coefficients to obtain			
			$a = 2,\ b = -7$			
	(iii)	Differentiates and uses given value of x to obtain c.	$y' = 2 - \dfrac{(c-5)}{(x-1)^2} = 0$	M1A1		
			When $x = 2$ then $c = 7$	A1	3	
	(iv)	Forms quadratic in x.	Let $y = \dfrac{2x^2 - 7x + 7}{(x-1)} = k$			
			$\Rightarrow 2x^2 - (7+k)x + 7 + k = 0$	B1		
		Uses discriminant.	No real roots $\Rightarrow (7+k)^2 - 8(7+k) < 0$	M1		
			$\Rightarrow k^2 + 6k - 7 < 0$	A1		
			$\Rightarrow (k+7)(k-1) < 0$			
		Obtains required result.	$\Rightarrow -7 < k < 1$	A1	4	**[11]**

2. Rational functions and graphs

Q10.

7	Vertical asymptote.	$x = 2$	B1	3	
	Divides by $(x-2)$	$y = x + p + 2 + \dfrac{2p+5}{x-2}$	M1		
	Oblique asymptote.	$y = x + p + 2$	A1	3	
	Differentiates.	$\dfrac{dy}{dx} = \dfrac{x^2 - 4x + 4 - 2p - 5}{(x-2)^2}$	M1A1		
		$y' = 0 \Rightarrow x^2 - 4x - (2p+1) = 0$	M1		
		$B^2 - 4AC > 0 \Rightarrow 16 + 4(2p+1) > 0$	M1		
		$\Rightarrow p > -\dfrac{5}{2}$	A1	5	
	Sketches graph. Working to show either $x^2 - x + 1 = 0$ has no real roots, or maximum value.	Axes and (0,–0.5) marked.. Upper Branch with minimum. Lower with maximum below x-axis. (Deduct at most 1 for poor forms at infinity.)	B1 B1 B1	3	**[11]**

Q11.

10	Intersections with axes.	(–1,0) , (2,0) (0,–1)	B1 B1	2	
		$yx^2 + 5xy + 10y = 5x^2 - 5x - 10$			
	Rearranges as a quadratic equation.	$(y-5)x^2 + (5y+5)x + 10(y+1) = 0$			
	Uses discriminant.	For real x $b^2 - 4ac \geq 0$			
		$\Rightarrow (5y+5)^2 - 40(y-5)(y+1) \geq 0 \ldots$	M1A1		
	Solves inequality.	$\Rightarrow (y-15)(y+1) \leq 0 \Rightarrow -1 \leq y \leq 15$ (AG)	M1A1	4	
	Finds turning points.	$y = -1 \Rightarrow x = 0$ $y = 15 \Rightarrow x = -4$	M1A1 A1		
		Turning points are (-4,15) and (0,–1)			
		$y = 5$.	B1		
	States asymptote.	Axes and asymptote correct	B1		
	Sketches graph.	Graph correct.	B1B1	7	**[13]**

2. Rational functions and graphs

Q12.

9	Forms quadratic equation in x.	$yx^2 + 2y = 2x^2 + 2x + 3$ $\Rightarrow (y-2)x^2 - 2x + (2y-3) = 0$	M1 A1		
	Uses discriminant to obtain condition for real roots.	For real x $4 - 4(y-2)(2y-3) \geq 0$ $\Rightarrow (2y-5)(y-1) \leq 0$ $\Rightarrow 1 \leq y \leq \frac{5}{2}$ (AG)	M1 A1	4	
	Differentiates and equates to zero. Solves equation.	$y' = 0$ $\Rightarrow (x^2+2)(4x+2) - 2x(2x^2+2x+3) = 0$ $\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1$ or $x = 2$ (Or substitutes $y = 1$ and $\frac{5}{2}$ in equation of C.)	M1		
	States coordinates of turning points.	Turning points are (–1, 1) and $\left(2, 2\frac{1}{2}\right)$	A1A1	3	
	Expresses y in an appropriate form. (May alternatively divide numerator and denominator by x^2.)	$y = 2 + \frac{2x-1}{x^2+2}$ As $x \to \pm\infty$ $y \to 2$ $\therefore y = 2$	M1 A1	2	
	Finds y-intercept and intersection with $y = 2$. Completes graph.	Shows $\left(0, 1\frac{1}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$ Completely correct graph.	B1 B1	2	**[11]**

Q13.

6	States vertical asymptote. Finds oblique asymptote.	Vertical asymptote is $x = 2$. $y = x + 2 + \frac{4}{x-2}$ Oblique asymptote is $y = x + 2$.	B1 M1 A1	3	
	Differentiates and equates to zero. Finds x coordinates. States coordinates of turning points.	$y' = 1 - \frac{4}{(x-2)^2} = 0 \Rightarrow (x-2)^2 = 4$ $x = 0, 4$. Turning points are (0,0) and (4,8)	M1 A1 A1	3	
	Deduct at most 1 mark for poor forms at infinity.	Axes and both asymptotes correct. Upper branch correct. Lower branch correct.	B1 B1 B1	3	**[9]**

2. Rational functions and graphs

Q14.

7	Finds asymptotes to C.	Vertical asymptote $x = 2$.	B1		
		$y = \lambda x + 1 + \dfrac{2}{x-2} \Rightarrow y = \lambda x + 1$ is oblique asymptote.	M1A1	3	
	Differentiates and equates to 0.	$y' = \lambda - 2(x-2)^{-2} = 0$ for turning points.	M1A1		
		$\lambda = \dfrac{2}{(x-2)^2} > 0 \Rightarrow$ no turning points if $\lambda < 0$.	A1	3	
		Or $y' = 0 \Rightarrow \lambda x^2 - 4\lambda x + 4\lambda - 2 = 0$	(M1A1)		
		Uses discriminant to show $8\lambda < 0 \Rightarrow$ no T.P.s.	(A1)		
	Sketch of graph. Deduct 1 mark for poor forms at infinity. Deduct 1 mark if intersections with axes not shown.	Axes and asymptotes.	B1		
		LH branch. RH branch. (Indicating intersections with axes at (0,0) and (3,0).)	B1B1	3	**[9]**

Q15.

9	States vertical asymptote.	Vertical asymptote is $x = 2$.	B1		
	Divides and states oblique asymptote.	$\Rightarrow$ oblique asymptote is $y = x - 1$.	M1 A1	3	
	Rearranges as quadratic in x.	$xy - 2y = x^2 - 3x + 3 \Rightarrow x^2 - (y+3)x + (3+2y) = 0$	B1		
	Uses discriminant to obtain condition stated.	For real x, $B^2 - 4AC \geq 0$			
		$\therefore (y+3)^2 - 4(3+2y) \geq 0$	M1		
		$\Rightarrow \ldots \Rightarrow (y-3)(y+1)) \geq 0$	A1		
		$\Rightarrow y \leq -1$ or $y \geq 3$			
		$\therefore$ no points for $-1 < y < 3$ (AG)	A1	4	
	Differentiates, puts = 0 and obtains x-values.	$y' = 1 - (x-2)^{-2} = 0 \Rightarrow x = 1$ or 3			
		(Or uses $y = -1$ and 3 to obtain x-values.)	M1		
	States stationary points.	Stationary points are (1,–1) and (3,3)	A1A1	3	
	Sketch.	One mark for each branch correctly placed.	B1B1	2	**[12]**

2. Rational functions and graphs

Q16.

10	States asymptotes.	Vertical: $x=1$ and Horizontal: $y=2$	B1B1	2	
	Obtains quadratic form in x.	$yx^2-2yx+y=2x^2-3x-2$ $\Rightarrow (y-2)x^2-(2y-3)x+(y+2)=0$	M1A1		
	Uses $B^2-4AC\geq 0$ for real roots.	For real x $(2y-3)^2-4(y-2)(y+2)\geq 0$ $\Rightarrow 12y\leq 25\Rightarrow y\leq \frac{25}{12}$.	M1 A1	4	
	Finds condition for $y'=0$.	$y'=0\Rightarrow$ $(x^2-2x+1)(4x-3)-(2x^2-3x-2)(2x-2)=0$	M1		
	Solves	$\Rightarrow x^2-8x+7=0\Rightarrow (x-7)(x-1)=0$ $\Rightarrow x=7$, (since $x=1$ is vertical asymptote).	A1		
	Obtains stationary point.	Stationary point is $\left(7,\frac{25}{12}\right)$	A1	3	
	Sketch showing:	Axes and asymptotes (-0.5,0) , (2,0) , (0,–2) and (4,2) Left hand branch. Right hand branch.	B1 B1 B1 B1	4	**[13]**

Q17.

10 (i)	Vertical asymptote. Oblique asymptote.	Asymptotes : $x=-1$ $y=px+4-p+(p-3)(x+1)^{-1}\Rightarrow y=px+4-p$	B1 M1A1	3	
(ii)	Obtains value of p. Sketches curve.	$p=4\Rightarrow$ x-axis is a tangent Correct location of turning points and asymptotes. Each branch.	B1 B1 B1B1	4	
(iii)	Proves required result. Sketches graph.	$p=1\Rightarrow y=x+3-2(x+1)^{-1}\Rightarrow y'=1+2(x+1)^{-2}$ $(\geqslant 1)$ Intersections on x-axis at $\left(-2\pm\sqrt{3},0\right)$ Each branch.	M1A1 B1 B1B1	5	**[12]**

2. Rational functions and graphs

Q18.

7	States vertical asymptote	Vertical asymptote is $x = -2$	B1		
	Expresses y in a suitable form	$y = 2x + 1 - 3(x+2)^{-1}$	M1		
	States oblique asymptote	Oblique asymptote is $y = 2x + 1$	A1	**(3)**	
	Differentiates wrt x	$y' = 2 + 3(x+2)^{-2}$	M1A1		
	and explains result	$(x+2)^{-2} > 0 \Rightarrow y' > 2$ (AG)	A1	**(3)**	
	Sketches graph.	Axes and asymptotes	B1		
	(Deduct 1 mark for poor forms at infinity)	Each branch, showing intersection with axes.	B1B1	**(3)**	**[9]**

Q19.

12O (i)	$d = -2$	B1 [1]
(ii)	$x = 0 \Rightarrow y = -2 = \frac{c}{d} \Rightarrow c = 4$	M1A1
	$y = ax + (b + 2a) + \frac{4 + 4a + 2b}{x - 2}$	M1A1
	Oblique asymptote $y = x + 1 \Rightarrow a = 1,\ b + 2a = 1 \Rightarrow b = -1$	A1A1 [6]
(iii)	$y = \frac{x^2 - x + 4}{x - 2} \Rightarrow x^2 - (y+1)x + 2y + 4 = 0$	M1
	For real x $B^2 - 4AC > 0 \Rightarrow (y+1)^2 - 4(2y+4) > 0$	M1
	$\Rightarrow y^2 - 6y - 15 > 0$	A1
	$\Rightarrow (y-3)^2 > 24$ (or for solving $y^2 - 6y - 15 = 0$)	M1A1
	$\Rightarrow y - 3 < -2\sqrt{6}$ or $y - 3 > -2\sqrt{6}$ (or for thumbnail sketch)	M1
	$\Rightarrow y < 3 - 2\sqrt{6}$ or $y > 3 + 2\sqrt{6}$ (AG)	A1
	(Note: if done by differentiation, max and min must be demonstrated for full marks.)	[7]

2. Rational functions and graphs

Q20.

11	**EITHER** $2+\frac{A}{x-1}+\frac{B}{x+1}=\frac{2(x^2-1)+A(x+1)+B(x-1)}{x^2-1}=\frac{2x^2+(A+B)x+A-B-2}{x^2-1}=\frac{2x^2-x+5}{x^2-1}$ $\Rightarrow A=3\ ,\ B=-4$ $y'=0\Rightarrow(x^2-1)(4x-1)-(2x^2-x+5).2x=0\Rightarrow x^2-14x+1=0$ $B^2-4AC=(-14)^2-4\times1\times1=192>0\Rightarrow$ two distinct turning points. Asymptotes are $x=1\ ,\ x=-1\ :\ y=2$ $y=2\Rightarrow 2x^2-x+5=2x^2-2\Rightarrow x=7\Rightarrow(7,\ 2)$ (Accept if labelled on graph.) Sketch: Middle branch crossing y-axis at $(0,-5)$ and left branch. Right branch. Working to show no intersections with x-axis.	M1 A1A1 [3] M1A1 M1A1 [4] B1B1 M1A1 B1 B1 B1 [7]

Q21.

4	Vertical asymptote is $x=1$ $y=2x+3+2(x-1)^{-1}\Rightarrow y=2x+3$ is the oblique asymptote. $2x^2+(1-y)x+(y-1)=0$ has real roots $\Leftrightarrow(1-y)^2-8(y-1)\geqslant 0$ $\Leftrightarrow y^2-10y+9\geqslant 0$ $\Leftrightarrow(y-1)(y-9)\geqslant 0$ Hence $(y-1)(y-9)<0\Rightarrow$ no real roots. i.e. $1<y<9\Rightarrow$ no points on C. (AG) Thumbnail sketch, or similar, required.	B1 M1A1 (3) M1A1 M1 A1 (4) **[7]**

2. Rational functions and graphs

Q22.

10	$-\frac{1}{2}+\frac{(3x-1)^2}{2(x^2+1)}=\frac{-x^2-1+9x^2-6x+1}{2(x^2+1)}=\frac{8x^2-6x}{2(x^2+1)}=\frac{4x^2-3x}{x^2+1}$ $\frac{9}{2}-\frac{(x+3)^2}{2(x^2+1)}=\frac{9x^2+9-x^2-6x-9}{2(x^2+1)}=\frac{8x^2-6x}{2(x^2+1)}=\frac{4x^2-3x}{x^2+1}$	M1A1 A1 (3)
	$\frac{(3x-1)^2}{2(x^2+1)}\geqslant 0\Rightarrow y\geqslant -\frac{1}{2}$	B1
	$\frac{(x+3)^2}{2(x^2+1)}\geqslant 0\Rightarrow y\leqslant \frac{9}{2}\Rightarrow -\frac{1}{2}\leqslant y\leqslant \frac{9}{2}$ (Not hence gets B1 only.)	B1 (2)
	Turning points occur at equality, so $\left(-3,\ \frac{9}{2}\right)$ and $\left(\frac{1}{3},\ -\frac{1}{2}\right)$	B1B1 (2)
	Asymptote is $y=4$.	B1 (1)
	Intersection with axes at (0, 0) and $\left(\frac{3}{4},\ 0\right)$, Intersection with asymptote at $\left(-\frac{4}{3},\ 4\right)$ Correct graph.	B1B1 B1 (3) **Total 11**

Q23.

8	$y'=0\Rightarrow (x+1)(4x+k)-\left(2x^2+kx\right)\times 1=0$ $\Rightarrow 4x^2+(4+k)x+k-2x^2-kx=0\Rightarrow 2x^2+4x+k=0$	**M1** **A1**
	$B^2-4AC<0\Rightarrow$ no stationary points $\Rightarrow 16-8k<0$ $\Rightarrow\ k>2$ for no stationary points.	**M1A1** **A1** [5]
	When $k=4$: Vertical asymptote: $x=-1$	**B1**
	Oblique asymptote: $y=2x+2-\frac{2}{x+1}\Rightarrow\ y=2x+2$	**M1A1**
	Axes and asymptotes Each branch.	**B1** **B1B1** [6] **Total 11**

2. Rational functions and graphs

Q24.

7	Vertical asymptote is $x = 2$.	**B1**
	$y = x + 2 + \dfrac{4}{x-2} \Rightarrow$ Oblique asymptote is $y = x + 2$.	**M1A1** [3]
	$y = \dfrac{x^2}{x-2} \Rightarrow x^2 - yx + 2y = 0$	**B1**
	Quadratic has no real roots (i.e. no points on C) if $\Delta < 0 \Rightarrow y^2 - 8y < 0$	**M1**
	$\Rightarrow y(y-8) < 0 \Rightarrow 0 < y < 8$. (**AG**) Correct inequality	**M1A1** [4]
	Axes and asymptotes. Each branch, showing (0,0) and (4,8). (Deduct at most 1 mark for poor forms at infinity and/or missing coordinates.)	**B1** **B1B1** [3]

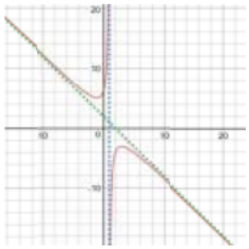
Q25.

5	$\dfrac{dy}{dx} = \dfrac{(x^2-9) - 2x(x+2)}{(x^2-9)^2}$	**B1**
	$= \dfrac{-x^2-4x-9}{(x^2-9)^2} = \dfrac{-(x+2)^2-5}{(x^2-9)^2} \Rightarrow \dfrac{dy}{dx} < 0$	**M1A1** [3]
	Asymptotes: $x = \pm 3$; $y = 0$	**B1B1** [2]
	Sketch: Axes and asymptotes ; Outside branches Middle branch, showing $\left(0, -\dfrac{2}{9}\right)$ and $(-2, 0)$. (Deduct at most 1 mark for poor forms at infinity.)	**B1B1** **B1** [3]

2. Rational functions and graphs

Q26.

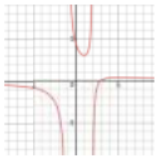
9(i)	$x=1$	**B1**	
	$y=2-x$	**M1A1**	
	Total:	**3**	
9(ii)	$y'=-1+4(1-x)^{-2}=0$	**M1**	
	$\Rightarrow x=-1,3$	**A1**	
	Turning points are $(-1,5)$ and $(3,-3)$	**A1**	
	Total:	**3**	
9(iii)	(0 , 6)	**B1**	
	$y=0\Rightarrow x^2-3x+6$; $\Delta=9-24\Rightarrow$ No intersection with x-axis	**B1**	
	Total:	**2**	

9(iv)		**B1 FT**	Asymptotes correct.
		B1B1	Each branch.
	Total:	**3**	

Q27.

9(i)	Degree of numerator < degree of denominator $\Rightarrow y=0$ is horizontal asymptote.	**B1**	
	$(x+1)(x-2)=0 \Rightarrow x=-1$ and $\Rightarrow x=2$ are vertical asymptotes.	**B1**	
		2	
9(ii)	$yx^2-(y+3)x+9-2y=0$	**M1**	
	No points on C if $(y+3)^2-4y(9-2y)<0$	**M1**	
	$\Rightarrow 9y^2-30y+9<0 \Rightarrow 3y^2-10y+3<0$	**A1**	
	$\Rightarrow (3y-1)(y-3)<0\Rightarrow \frac{1}{3}<y<3$	**A1**	AG
		4	
9(iii)	$\frac{dy}{dx}=0\Rightarrow 3(x^2-x-2)-(3x-9)(2x-1)=0$	**B1**	
	$\Rightarrow \ldots \Rightarrow (x-1)(x-5)=0$	**B1**	
	$\Rightarrow$ Turning points are (1,3) and $\left(5,\frac{1}{3}\right)$.	**B1**	
		3	

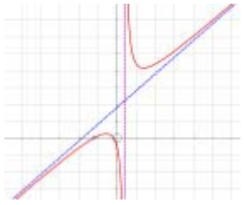
2. Rational functions and graphs

9(iv)		Axes, asymptotes and points on axes (0, 4.5) (3,0).	**B1**	
		RH branch ; Other two branches	**B1B1**	
			3	

Q28.

6(i)	Vertical asymptote is $x=-b$.	**B1**	
	$x^2+b=(x+b)(x-b)+b^2+b$ or $x+b\overline{)x^2+0x+b}$ $\quad x-b$	**M1**	By inspection or long division.
	Thus the oblique asymptote is $y=x-b$	**A1**	
		3	
6(ii)	If $y=0$ then $x^2+b=0$ which has no real root.	**B1**	Must refer to b > 0 OE
		1	
6(iii)	$\frac{dy}{dx}=\frac{2x(x+b)-(x^2+b)}{(x+b)^2}=0\Rightarrow x^2+2bx-b=0$ Or differentiating $y=x-b+\frac{b^2+b}{x+b}$ and setting $\frac{dy}{dx}=0$ gives $1-\frac{b^2+b}{(x+b)^2}=0$.	**M1**	Find $\frac{dy}{dx}$ and set = 0
	$b^2+b>0$ Therefore there are two stationary points on C	**A1**	Use discriminant or $(x+b)^2$ to show two stationary points
		2	

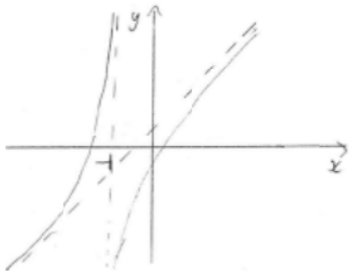
Q29.

4(i)	$(-6,0),(-1,0)$	**B1**	States points of intersection with x-axis.
	$(0,-3)$	**B1**	States y-intercept
4(ii)	One asymptote is $x=2$.	**B1**	
	$y=x+9+\frac{24}{x-2}\Rightarrow$ other asymptote is $y=x+9$.	**M1 A1**	By inspection or long division. A0 if error in division
4(iii)		**B1**	Sketches axes and asymptotes, labelled or to scale
		B1	Upper branch correctly located and orientated.
		B1	Lower branch correctly located and orientated. Penalise at most 1 mark for poor forms at infinity
		8	

2. Rational functions and graphs

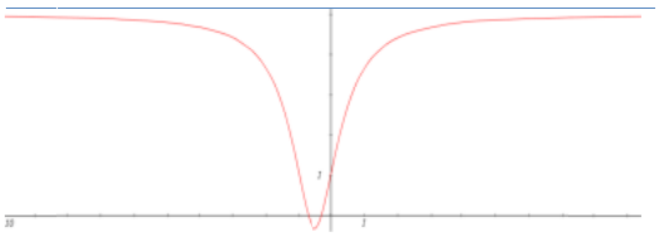
Q30.

6(i)	Vertical asymptote is $x=-1$.	**B1**	
	$x^2+ax-1=(x+1)(x+a-1)-a$ or $x+1\overline{)x^2+ax-1}$ $x+a-1$	**M1**	By inspection or long division.
	Thus the oblique asymptote is $y=x+a-1$	**A1**	
		3	
6(ii)	$a^2+4>0$	**B1**	
		1	
6(iii)	$\frac{(x+1)(2x+a)-(x^2+ax-1)}{(x+1)^2}=0\Rightarrow x^2+2x+a+1=0$ Discriminant $=4-4(a+1)<0$	**M1**	
	Therefore there are no stationary points on C.	**A1**	
		2	

6(iv)		**B1**	Correct y-intercept and asymptotes drawn.
		B1 B1	Each branch correct
		3	

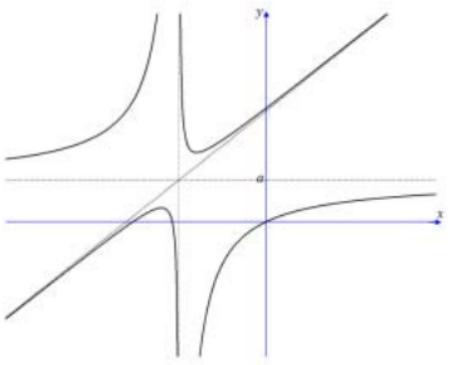
Q31.

9(i)	$y=5-\frac{4}{x^2+x+1}$	**M1**	Alt method: Finding limit
	As $x\to\pm\infty$, $y\to 5\therefore y=5$ CAO	**A1**	
		2	

9(ii)	$yx^2+yx+y=5x^2+5x+1$ $\Rightarrow(y-5)x^2+(y-5)x+(y-1)=0$	**B1**	Forms quadratic equation in x.
	For real x, $(y-5)^2-4(y-5)(y-1)\geqslant 0$ (condone >)	**M1**	Uses discriminant
	$\Rightarrow(y-5)(3y+1)\leqslant 0$	**M1**	Factorising
	$\Rightarrow -\frac{1}{3}\leqslant y<5$, because $y=5$ is an asymptote (www)	**A1**	Explaining strict upper inequality (AG)
		4	
9(iii)	$y'=0\Rightarrow(x^2+x+1)(10x+5)-(5x^2+5x+1)(2x+1)=0$	**M1**	Differentiates and equates to 0.
	$\Rightarrow 4(2x+1)=0\Rightarrow x=-\frac{1}{2}, y=-\frac{1}{3}$	**A1**	
		2	
9(iv)		**B1FT**	Positive y-intercept at (0,1), FT dep on minimum point from **(iii)**.
		B1	Correct asymptote and completely correct graph.
		2	

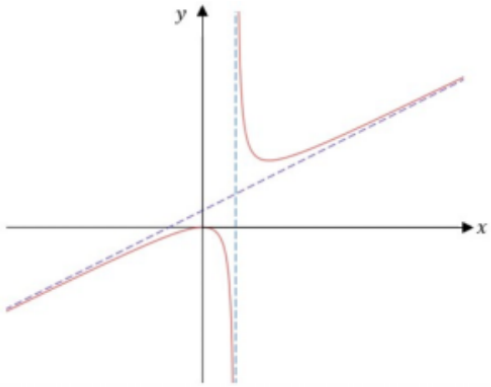
Q32.

10(i)	$x=-5$ and $y=a$	**B1 B1**	
		2	
10(ii)	$x^2+(a+10)x+5a+26=(x+5)(x+a+5)+1$	**M1**	By inspection or long division.
	oblique asymptote is $y=x+a+5$	**A1**	
		2	
10(iii)	$x^2+10x+5a+26=0$	**M1**	Puts y–values equal and forms quadratic equation.
	$10^2-4(5a+26)=-4-20a<0$ so no intersection point	**A1**	Correct discriminant and conclusion.
		2	
10(iv)	$(x+5)(2x+a+10)-x^2-ax-10x-5a-26=0$	**M1**	Differentiates and forms quadratic equation.
	$x^2+10x+24=0$	**A1**	
	Stationary points are $(-4,a+2)$ and $(-6,a-2)$	**A1**	Must have both points.
		3	

10(v)		**B1**	Asymptotes drawn, intersection correct.
		B1	C_1 correct.
		B1	C_2 correct.
		3	

Q33.

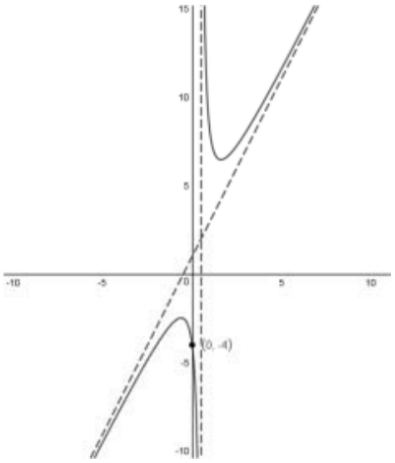
6(i)	$x=\frac{1}{k}$	**B1**	States vertical asymptote.
	$y=\frac{(kx-1)\left(k^{-1}x+k^{-2}\right)+k^{-2}}{kx-1}$	**M1**	Finds oblique asymptote.
	Oblique asymptote is $y=k^{-1}x+k^{-2}$	**A1**	
		3	
6(ii)	$y'=\frac{2x(kx-1)-kx^2}{(kx-1)^2}=0\Rightarrow kx^2-2x=0$	**M1**	Differentiates and equates to 0.
	$x=0,\ 2k^{-1}$	**A1**	Finds x-coordinates.
	$(0,0),\ \left(2k^{-1},4k^{-2}\right)$	**A1**	Finds y-coordinates
		3	

6(iii)		**B1**	Axes and asymptotes correct.
		B1	Upper branch correct.
		B1	Lower branch correct. Deduct at most 1 mark for poor forms at infinity.
		3	

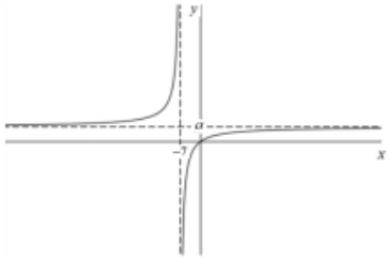
2. Rational functions and graphs

Q34.

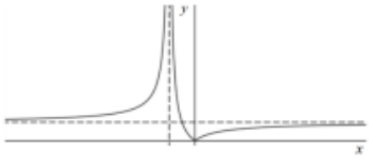
4(i)	$x^2+1=(ax+b)(2x+1)+c$	**M1**	Uses that $2x+1$ is the quotient.
	$\Rightarrow a=\frac{1}{2}, b=-\frac{1}{4}$	**A1 A1**	
		3	
4(ii)	$x=\frac{1}{2}$	**B1 FT**	
		1	

4(iii)		**B1**	Intersection (0,-4) given and asymptotes drawn.
		B1	Left branch correct.
		B1 FT	Right branch correct. Deduct at most one mark for poor forms at infinity.
		3	

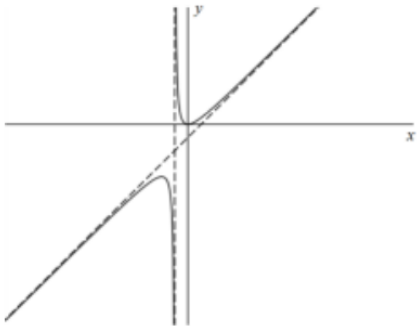
Q35.

1(a)		**B1**
	(**B1** for axes and asymptotes correct, **B1** for branches correct)	**B1**
		2

2. Rational functions and graphs

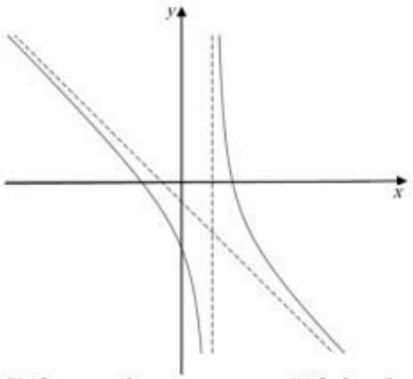
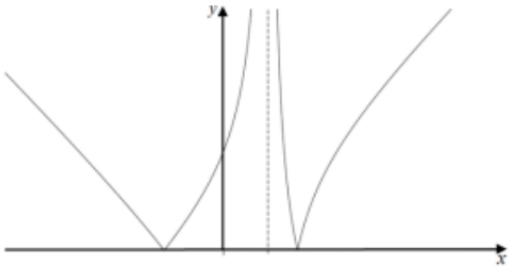
1(b)	 (**B1 FT** from sketch in part **(a)**)	**B1 FT**
	$\frac{ax}{x+7}=\frac{a}{2}$ or $\frac{ax}{x+7}=-\frac{a}{2}$	**M1**
	$x=7$ and $x=-\frac{7}{3}$	**A1**
	$x<-7$, $-7<x<-\frac{7}{3}$, $x>7$	**A1**
		4

Q36.

3(a)	$x=-\frac{1}{2}$	**B1**
	$y=\frac{(2x+1)(\frac{1}{2}x-\frac{1}{4})+\frac{1}{4}}{2x+1}=\frac{1}{2}x-\frac{1}{4}+\frac{1}{4(2x+1)}$	**M1**
	$y=\frac{1}{2}x-\frac{1}{4}$	**A1**
		3
3(b)	$\frac{dy}{dx}=\frac{1}{2}-\frac{1}{2(2x+1)^2}=0\Rightarrow(2x+1)^2=1$	**M1**
	$x=0,-1$	**A1**
	$(0,0),(-1,-1)$	**A1**
		3
3(c)	 (**B1** for axes and asymptotes correct, **B1** for upper branch correct, **B1** for lower branch correct)	**B1** **B1** **B1**
		3

2. Rational functions and graphs

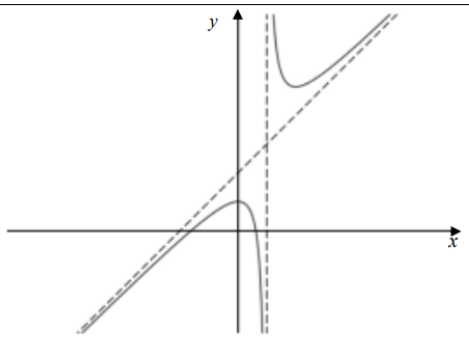
Q37.

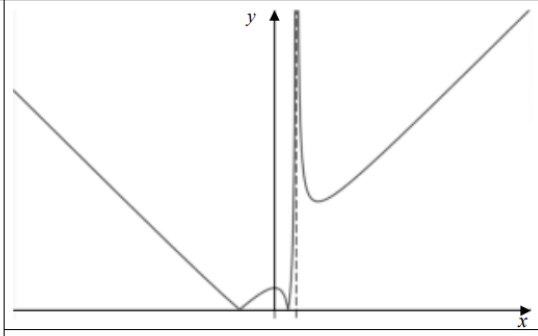
6(a)	$x=\frac{3}{2}$	**B1**
	$-2x^2+x+10=(2x-3)(-x-1)+7 \Rightarrow y=-x-1$	**M1 A1**
		3
6(b)	$\frac{dy}{dx}=\frac{(2x-3)(1-4x)-2(10+x-2x^2)}{(2x-3)^2}$	**M1**
	$4x^2-12x+23=0 \quad \left(\text{or } \frac{dy}{dx}=-1-\frac{14}{(2x-3)^2}\right)$	**A1**
	$12^2-4(4)(23)=-224<0$ (or $y'<0$) $\Rightarrow$ No turning points.	**A1**
		3
6(c)	 (**B1** for axes and asymptotes correct, **B1** for branches correct)	**B1** **B1**
	$(0,-\frac{10}{3}),(-2,0),(\frac{5}{2},0)$	**B1**
		3
6(d)	 (**B1 FT** for their sketch in **(c)**, **B1** for correct shape at infinity)	**B1 FT** **B1**
	$\frac{10+x-2x^2}{2x-3}=4$ or $\frac{10+x-2x^2}{2x-3}=-4$ $2x^2+7x-22=0$ or $2x^2-9x+2=0$	**M2**
	$x=-\frac{11}{2},2$ or $x=\frac{1}{4}(9\pm\sqrt{65})$	**A1**
	$-\frac{11}{2}<x<\frac{1}{4}(9-\sqrt{65})$ and $2<x<\frac{1}{4}(9+\sqrt{65})$	**A1 FT**
		6

2. Rational functions and graphs

Q38.

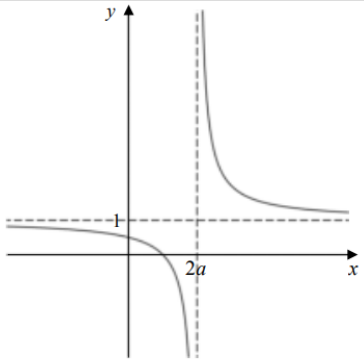
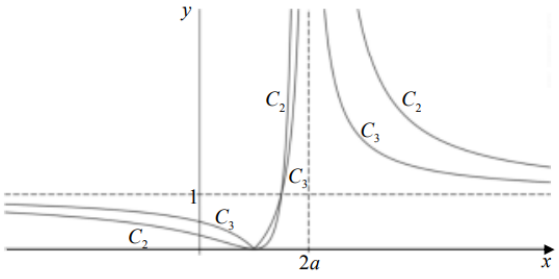
6(a)	$x=1$	**B1**	States vertical asymptote.
	$x^2+x-1=(x-1)(x+2)+1\Rightarrow y=x+2$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx-y=x^2+x-1\Rightarrow x^2+(1-y)x+y-1=0$	**M1 A1**	Forms quadratic in x.
	$(1-y)^2-4(y-1)<0\Rightarrow y^2-6y+5<0$	**M1**	Uses that discriminant is negative if there are no values of x.
	$1<y<5$	**A1**	AG
		4	

6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
	$(0,1)$, $\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5},0\right)$, $\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5},0\right)$	**B1**	States coordinates of intersections with axes, can be labelled on graph. Accept $(0.618,0)$ and $(-1.62,0)$.
		3	

6(d)		**B1 FT**	FT from sketch in **(c)** with both branches.
		B1	Correct shape at extremities.
		2	

2. Rational functions and graphs

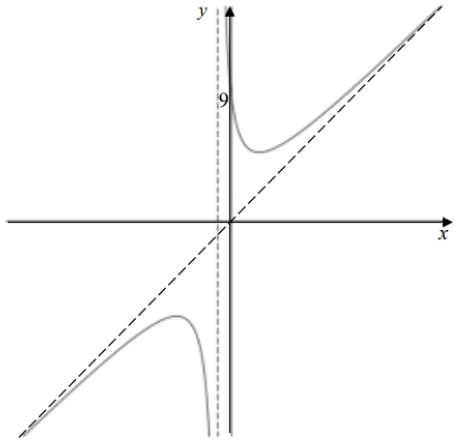
Q39.

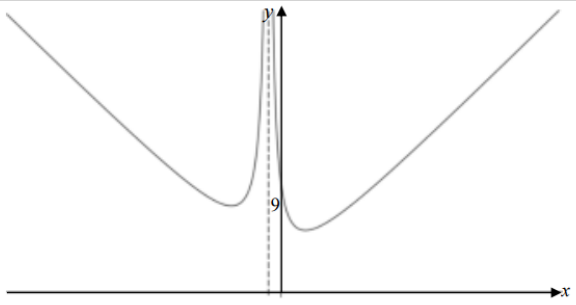
6(a)		**B1**	Asymptotes labelled.
		B1	Correct position and shape. Not too truncated.
		2	
6(b)(i)	$\frac{dy}{dx} = -\frac{2a(x-a)}{(x-2a)^3} = 0$	**M1**	Sets $\frac{dy}{dx} = 0$.
	Solves for x.	**M1**	
	Since $x \neq 2a$, stationary point is $(a, 0)$.	**A1**	
		3	
6(b)(ii)	$(x-a)^2 = (x-a)(x-2a)$ or $(x-a)^2 = -(x-a)(x-2a)$	**M1**	Finds a critical point.
	$x = a$ and $\frac{3}{2}a$ **or** $(a, 0)$ **or** $(\frac{3}{2}a, 1)$	**A1**	Both x values or one correct point.
	$(a, 0)$ and $(\frac{3}{2}a, 1)$	**A1**	
		3	
6(c)		**B1**	Axes and asymptotes.
		B1 FT	C_3 FT from their sketch in part **(a)**. Clearly identified.
		B1	Correct shape of C_2. Clearly identified.
		B1	Relative positions and intersections correct.
	$x \leqslant \frac{3}{2}a$	**B1**	Accept algebraic method.
		5	

2. Rational functions and graphs

Q40.

7(a)	$x=-1$	**B1**	States vertical asymptote.
	$y=\dfrac{x(x+1)+9}{x+1}=x+\dfrac{9}{x+1}$	**M1**	Finds oblique asymptote.
	$y=x$	**A1**	
		3	
7(b)	$\dfrac{dy}{dx}=1-9(x+1)^{-2}=0\Rightarrow(x+1)^2=9$	**M1 A1**	Differentiates and sets derivative equal to 0.
	$(2,5)$	**A1**	
	$(-4,-7)$	**A1**	
		4	

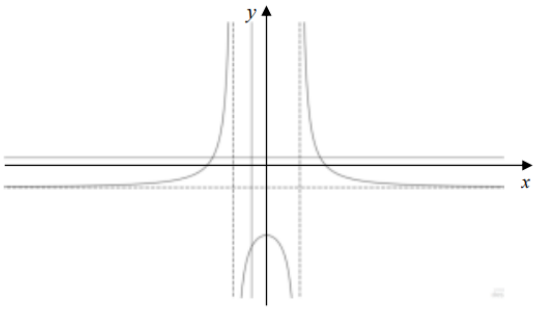
7(c)		**B1**	Axes labelled and correct asymptotes drawn.
		B1	Upper branch with $(0,9)$ stated or shown on diagram.
		B1	Lower branch correct and good approach to asymptotes throughout, no extra branches.
		3	

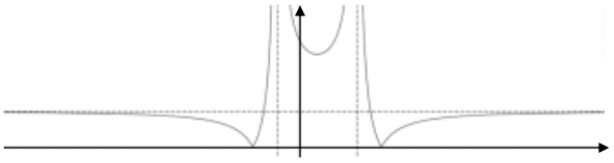
7(d)		**B1 FT**	FT from sketch in **(c)** with asymptotes shown.
	$x^2+x+9=\dfrac{13}{2}(x+1)$ or $x^2+x+9=-\dfrac{13}{2}(x+1)$ $x^2-\frac{11}{2}x+\frac{5}{2}=0$ or $x^2+\frac{15}{2}x+\frac{31}{2}=0$	**M1 M1**	Finds critical points, award M1 for each case. May state that $x^2+x+9=-\frac{13}{2}(x+1)$ has no real solutions since $7>\frac{13}{2}$.
	$x=\frac{1}{2}, 5$	**A1**	
	$x<\frac{1}{2}$ and $x>5$.	**A1**	
		5	

2. Rational functions and graphs

Q41.

7(a)	$x=\frac{1-\sqrt{5}}{2}, x=\frac{1+\sqrt{5}}{2}$	**B1**	Vertical asymptotes. Must be exact.
	$y=-1$	**B1**	Horizontal asymptote.
		2	
7(b)	$\frac{dy}{dx}=\frac{(1+x-x^2)(2x-1)-(x^2-x-3)(1-2x)}{(1+x-x^2)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$(2x-1)(-2)=0$	**M1**	Sets their $\frac{dy}{dx}$ equal to 0 and forms equation.
	$\left(\frac{1}{2},-\frac{13}{5}\right)$	**A1**	WWW
		3	

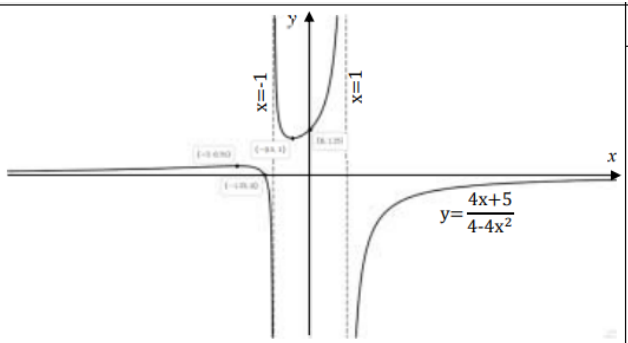
7(c)		**B1**	Both axes labelled and correct asymptotes shown.
		B1	Correct shape and position, with all asymptotic behaviour clear.
	$\left(\frac{1}{2}+\frac{1}{2}\sqrt{13},0\right)$, $\left(\frac{1}{2}-\frac{1}{2}\sqrt{13},0\right)$, $(0,-3)$	**B1**	States exact coordinates of intersections with axes.
		3	

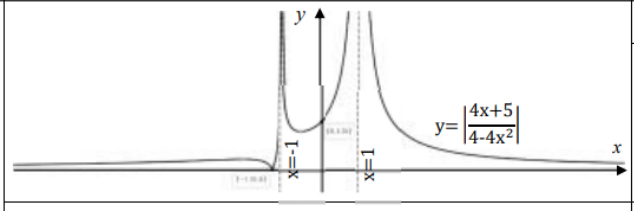
7(d)		**B1 FT**	FT from sketch in **(c).**
		B1	Correct shape as x tends to infinity and intersections with x axis.
	$\frac{x^2-x-3}{1+x-x^2}=3$ or $\frac{x^2-x-3}{1+x-x^2}=-3$ $4x^2-4x-6=0$ or $-2x^2+2x=0$	**M2**	Finds critical points, award M1 for each case.
	$x=\frac{1}{2}+\frac{1}{2}\sqrt{7}$, $x=\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $x=0$, $x=1$	**A1**	Must be exact.
	$x<\frac{1}{2}-\frac{1}{2}\sqrt{7}$, $0<x<1$, $x>\frac{1}{2}+\frac{1}{2}\sqrt{7}$	**A1 FT**	Must be three distinct regions and strict inequalities.
		6	

2. Rational functions and graphs

Q42.

7(a)	$x=1,\ x=-1$	**B1**	Vertical asymptotes.
	$y=0$	**B1**	Horizontal asymptote.
		2	
7(b)	$\frac{dy}{dx}=\frac{(4-4x^2)(4)-(4x+5)(-8x)}{(4-4x^2)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$16x^2+40x+16=0$	**M1**	Sets equal to 0 and forms equation.
	$(-2,\frac{1}{4}),\ (-\frac{1}{2},1)$	**A1 A1**	
		4	

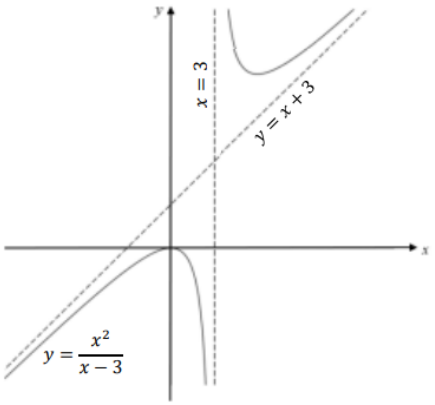
7(c)		**B1**	Axes and asymptotes.
		B1	Correct shape and position.
	$(-\frac{5}{4},0),\ (0,\frac{5}{4})$	**B1**	States coordinates of intersections with axes.
		3	

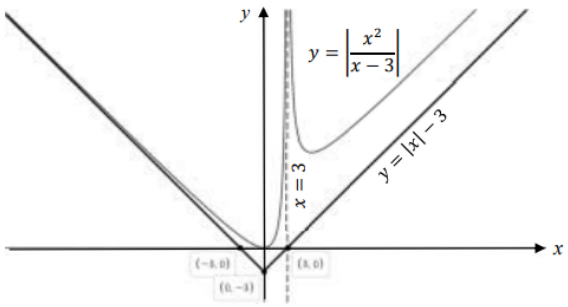
7(d)		**B1 FT**	FT from sketch in part **(c)**.
		B1	Correct shape at infinity.
	$\frac{4x+5}{4-4x^2}=\frac{5}{4}$ or $\frac{4x+5}{4-4x^2}=-\frac{5}{4}$ $5x^2+4x=0$ or $5x^2-4x-10=0$	**M2**	Finds critical points, award M1 for each case.
	$x=-\frac{4}{5},\ x=0$ or $x=\frac{2}{5}-\frac{3}{5}\sqrt{6},\ x=\frac{2}{5}+\frac{3}{5}\sqrt{6}$	**A1**	A0 if -1.07, 1.87
	$\frac{2}{5}-\frac{3}{5}\sqrt{6}<x<-\frac{4}{5},\ 0<x<\frac{2}{5}+\frac{3}{5}\sqrt{6}$, $x\neq\pm1$	**A1 FT**	Condone exclusion of $x=\pm1$ from the range.
		6	

2. Rational functions and graphs

Q43.

6(a)	$x=3$	**B1**	States vertical asymptote.
	$y=x+3+\frac{9}{x-3}$ leading to $y=x+3$	**M1 A1**	Finds oblique asymptote.
		3	
6(b)	$yx-3y=x^2$ leading to $x^2-yx+3y=0$	**M1 A1**	Forms quadratic in x.
	$y^2-4(3y)<0$ leading to $y^2-12y<0$	**M1**	Uses that discriminant is negative.
	$0<y<12$	**A1**	AG
		4	

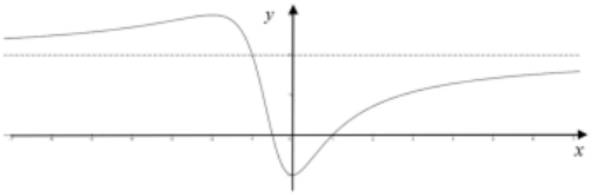
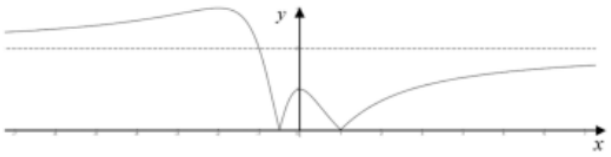
6(c)		**B1**	Axes and asymptotes.
		B1	Branches correct.
		2	

6(d)(i)		**B1 FT**	FT from sketch in **(c)**.
		B1	Correct shape at infinity.
		B1	Correct shape of $y=\|x\|-3$.
		B1	Correct intercepts with axes (may be seen on graph).
		4	
6(d)(ii)	$c \leqslant -3$	**B1**	
		1	

2. Rational functions and graphs

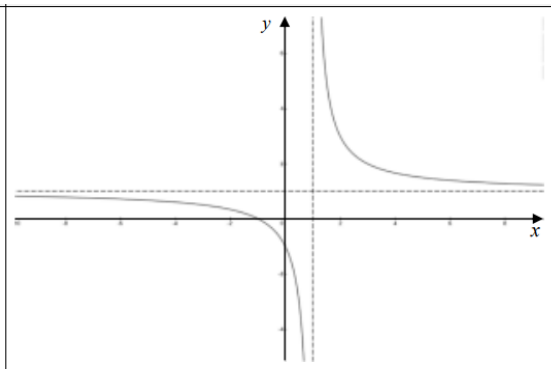
Q44.

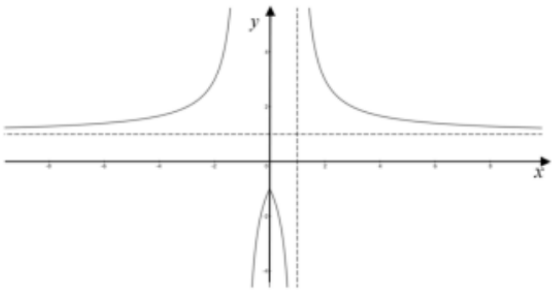
5(a)	$-3<0$	**M1 A1**	Finds discriminant or roots of x^2+x+1, or completes square.
	$y=2$	**B1**	Horizontal asymptote.
		3	
5(b)	$\frac{dy}{dx}=\frac{(x^2+x+1)(4x-1)-(2x^2-x-1)(2x+1)}{(x^2+x+1)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$3x^2+6x=0$	**M1**	Sets equal to 0 and forms equation.
	$(0,-1)$ $(-2,3)$	**A1 A1**	One point correct, or both x values. Other point correct.
	Alternative method for question 5(b)		
	$2x^2-x-1-y(x^2+x+1)=0$	**M1**	Forms quadratic equation
	Finds discriminant $(y+1)^2-4(y+1)(y-2)$ AND states y exists if discriminant ≥ 0 OR does not exist if discriminant <0	**M1**	
	Finds (0, -1) and (-2, 3)	**A1**	
	Explains why they are stationary values.	**A1**	Double x roots for $y=-1$ and $y=3$ or no vertical asymptote etc
		4	

5(c)		**B1**	Axes and asymptote.
		B1	Correct shape and position.
	$(1,0)$, $(-\frac{1}{2},0)$, $(0,-1)$	**B1**	States coordinates of intersections with axes.
		3	
5(d)		**B1 FT**	FT from sketch in part **(c)**. There must be cusps on the x axis.
	$0<k<1$	**B1**	
		2	

2. Rational functions and graphs

Q45.

1(a)		**B1**	Axes and asymptotes.
		B1	Branches correct.
		2	

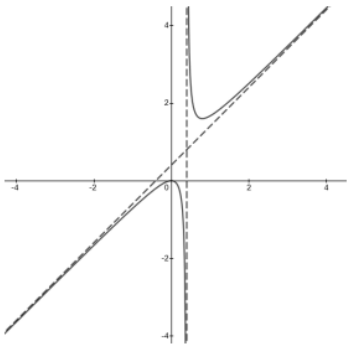
1(b)		**B1 FT**	FT from sketch in part **(a)**, symmetrical about $x = 0$.
		B1	Correct shape at $x = 0$ (reflection not turning point).
	$\frac{\lvert x\rvert+1}{\lvert x\rvert-1} = -2$ leading to $\lvert x\rvert = \frac{1}{3}$	**M1**	Finds critical point(s).
	$-1 < x < -\frac{1}{3}$, $\frac{1}{3} < x < 1$	**A1**	
		4	

Q46.

3(a)	$x = 1$	**B1**	States vertical asymptote.
	$y = \frac{(x-1)(ax+a+1)+a}{x-1} = ax+a+1+\frac{a}{x-1}$	**M1**	Finds oblique asymptote.
	$y = ax+a+1$	**A1**	
		3	

3(b)	$ax^2+x-1=yx-y$ leading to $ax^2-(y-1)x+(y-1)=0$	**M1 A1**	Forms a three term quadratic in x.
	$(y-1)^2-4a(y-1)<0$	**M1**	Correct inequality using discriminant.
	$1<y<1+4a$	**A1**	AG Clear method to reach given answer eg $(y-1)(y-1-4a)<0$
	Alternative method for question 3(b)		
	$ax^2+x-1=yx-y$ leading to $ax^2-(y-1)x+(y-1)=0$	**M1 A1**	Forms three term quadratic in x.
	$(y-1)^2-4a(y-1)\geq 0$	**M1**	Correct inequality using discriminant. Must be clear looking for where values of y exist.
	$1<y<1+4a$	**A1**	AG Clear method to reach given answer eg $(y-1)(y-1-4a)\geqslant 0$
	Second alternative method for question 3(b)		
	$\frac{dy}{dx}=\frac{(x-1)(2ax+1)-(ax^2+x-1)}{(x-1)^2}=0$	**M1**	
	$x=0$ or $x=2$	**A1**	
	Proves (0, 1) is local maximum and $(2,4a+1)$ is local minimum.	**M1**	Either $\frac{d^2y}{dx^2}$ or gradient either side
	Reference to position of asymptote or two distinct branches to justify $1<y<1+4a$	**A1**	AG
		4	

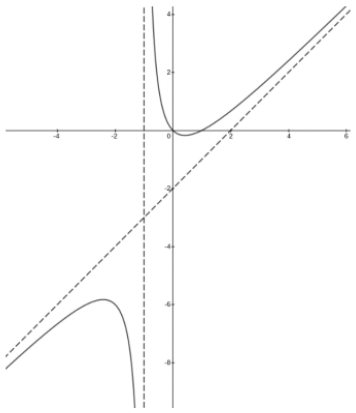
Q47.

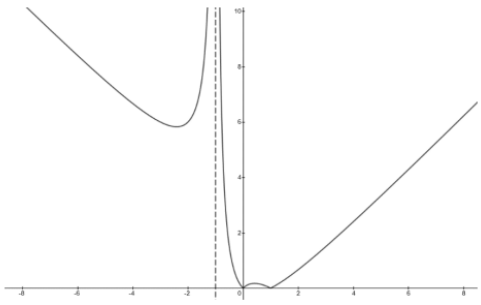
7(a)	$x=\frac{2}{5}$	**B1**	Vertical asymptote.
	$y=\frac{(5x-2)(x+\frac{2}{5})+\frac{4}{5}}{5x-2}$ leading to $y=x+\frac{2}{5}$	**M1 A1**	Oblique asymptote.
		3	
7(b)	$\frac{dy}{dx}=\frac{(5x-2)(10x)-(5x^2)(5)}{(5x-2)^2}$	**M1**	Finds $\frac{dy}{dx}$.
	$5x^2-4x=0$	**M1**	Sets equal to 0 and forms quadratic equation.
	$(0,0)$, $(\frac{4}{5},\frac{8}{5})$	**A1 A1**	
		4	
7(c)		**B1**	Axes and asymptotes.
		B1	Correct upper branch and asymptotic behaviour.
		B1	Correct lower branch.
		3	

2. Rational functions and graphs

Q48.

7(a)	$x=-1$	**B1**	States vertical asymptote.
	$y=\dfrac{(x+1)(x-2)+2}{x+1}$	**M1**	Finds oblique asymptote.
	$y=x-2$	**A1**	
		3	
7(b)	$\dfrac{dy}{dx}=1-2(x+1)^{-2}=0\Rightarrow(x+1)^2=2$	**M1 A1**	Differentiates and sets derivative equal to 0.
	$\left(-1+\sqrt{2},-3+2\sqrt{2}\right),\left(-1-\sqrt{2},-3-2\sqrt{2}\right)$	**A1 A1**	
		4	

7(c)		**B1**	Axes and asymptotes labelled.
		B1	Upper branch with $(0,0)$ and $(1,0)$ stated or clear on scale.
		B1	Lower branch correct and all asymptotic approaches correct.
		3	

7(d)		**B1 FT**	FT from sketch in part (c).
	$\dfrac{x^2-x}{x+1}=6$ or $\dfrac{x^2-x}{x+1}=-6$ $x^2-7x-6=0$ or $x^2+5x+6=0$	**M2**	Finds critical points, award M1 for each case.
	$x=-3,-2,\ \frac{7}{2}-\frac{1}{2}\sqrt{73},\ \frac{7}{2}+\frac{1}{2}\sqrt{73}$	**A1**	
	$-3<x<-2$ and $\frac{7}{2}-\frac{1}{2}\sqrt{73}<x<\frac{7}{2}+\frac{1}{2}\sqrt{73}$	**A1**	
		5	

3. Summation of series

Q1.

2 Given that

$$u_n = \ln\left(\frac{1+x^{n+1}}{1+x^n}\right),$$

where $x > -1$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

Find the sum to infinity of the series

$$u_1 + u_2 + u_3 + \ldots$$

when

(i) $-1 < x < 1$, [1]

(ii) $x = 1$. [1]

9231/01/M/J/08 Q2

Q2.

9 Use induction to prove that

$$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}.$$ [6]

Show that

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} < \frac{3}{8N^2}.$$ [4]

9231/01/O/N/08 Q9

Q3.

2 Verify that, for all positive values of n,

$$\frac{1}{(n+2)(2n+3)} - \frac{1}{(n+3)(2n+5)} = \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)}.$$ [2]

For the series

$$\sum_{n=1}^{N} \frac{4n+9}{(n+2)(n+3)(2n+3)(2n+5)},$$

find

(i) the sum to N terms, [3]

(ii) the sum to infinity. [1]

9231/01/M/J/09 Q2

3. Summation of series

Q4.

Prove by induction that

$$\sum_{n=1}^{N} n^3 = \tfrac{1}{4}N^2(N+1)^2. \qquad [5]$$

Use this result, together with the formula for $\sum_{n=1}^{N} n^2$, to show that

$$\sum_{n=1}^{N}(20n^3 + 36n^2) = N(N+1)(N+3)(5N+2). \qquad [3]$$

Let

$$S_N = \sum_{n=1}^{N}(20n^3 + 36n^2 + \mu n).$$

Find the value of the constant μ such that S_N is of the form $N^2(N+1)(aN+b)$, where the constants a and b are to be determined. [3]

Show that, for this value of μ,

$$5 + \frac{22}{N} < N^{-4}S_N < 5 + \frac{23}{N},$$

for all $N \geqslant 18$. [3]

9231/01/O/N/09 Q11

Q5.

4 The sum S_N is defined by $S_N = \sum_{n=1}^{N} n^5$. Using the identity

$$\left(n+\tfrac{1}{2}\right)^6 - \left(n-\tfrac{1}{2}\right)^6 \equiv 6n^5 + 5n^3 + \tfrac{3}{8}n,$$

find S_N in terms of N. [You need not simplify your result.] [4]

Hence find $\lim_{N\to\infty} N^{-\lambda}S_N$, for each of the two cases

(i) $\lambda = 6$,

(ii) $\lambda > 6$.

[3]

9231/11/M/J/10 Q4

3. Summation of series

Q6.

2 By considering the identity

$$\cos[(2n-1)\alpha] - \cos[(2n+1)\alpha] \equiv 2\sin\alpha\sin 2n\alpha,$$

show that if α is not an integer multiple of π then

$$\sum_{n=1}^{N} \sin(2n\alpha) = \tfrac{1}{2}\cot\alpha - \tfrac{1}{2}\operatorname{cosec}\alpha\cos[(2N+1)\alpha]. \quad [4]$$

Deduce that the infinite series

$$\sum_{n=1}^{\infty} \sin\left(\tfrac{2}{3}n\pi\right)$$

does not converge. [1]

9231/13/M/J/10 Q2

Q7.

2 Use the method of differences to find S_N, where

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+2)}. \quad [4]$$

Deduce the value of $\lim_{N\to\infty} S_N$. [1]

9231/01/O/N/10 Q2

Q8.

1 Express $\dfrac{1}{(2r+1)(2r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)}. \quad [4]$$

Deduce the value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)}. \quad [1]$$

9231/11/M/J/11 Q1

3. Summation of series

Q9.

1 Find $2^2 + 4^2 + \ldots + (2n)^2$. [2]

Hence find $1^2 - 2^2 + 3^2 - 4^2 + \ldots - (2n)^2$, simplifying your answer. [3]

9231/13/M/J/11 Q1

Q10.

1 Verify that $\frac{1}{n^2} - \frac{1}{(n+1)^2} = \frac{2n+1}{n^2(n+1)^2}$. [1]

Let $S_N = \sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2}$. Express S_N in terms of N. [2]

Let $S = \lim_{N\to\infty} S_N$. Find the least value of N such that $S - S_N < 10^{-16}$. [3]

9231/13/O/N/11 Q1

Q11.

3 Given that $f(r) = \frac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$ [2]

Hence find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$. [3]

Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

9231/11/M/J/12 Q3

Q12.

1 Find the sum of the first n terms of the series

$$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \ldots$$

and deduce the sum to infinity. [5]

9231/13/M/J/12 Q1

3. Summation of series

Q13.

4 Let $f(r) = r(r+1)(r+2)$. Show that

$$f(r) - f(r-1) = 3r(r+1). \quad [1]$$

Hence show that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [2]

Using the standard result for $\sum_{r=1}^{n} r$, deduce that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [2]

Find the sum of the series

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + 2(n-1)^2 + n^2,$$

where n is odd. [3]

9231/11/O/N/12 Q4

Q14.

1 Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

9231/13/O/N/12 Q1

Q15.

5 Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

9231/11/M/J/13 Q5

Q16.

1 Let $f(r) = r!(r-1)$. Simplify $f(r+1) - f(r)$ and hence find $\sum_{r=n+1}^{2n} r!(r^2+1)$. [5]

9231/13/M/J/13 Q1

3. Summation of series

Q17.

3 It is given that

$$S_n = \sum_{r=1}^{n} u_r = 2n^2 + n.$$

Write down the values of S_1, S_2, S_3, S_4. Express u_r in terms of r, justifying your answer. [4]

Find

$$\sum_{r=n+1}^{2n} u_r.$$ [3]

9231/11/O/N/13 Q3

Q18.

1 Express $\dfrac{1}{r(r+1)(r-1)}$ in partial fractions. [1]

Find

$$\sum_{r=2}^{n} \frac{1}{r(r+1)(r-1)}.$$ [4]

State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r(r+1)(r-1)}.$$ [1]

9231/13/O/N/13 Q1

Q19.

2 Expand and simplify $(r+1)^4 - r^4$. [1]

Use the method of differences together with the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [4]

9231/11/M/J/14 Q2

Q20.

2 Show that the difference between the squares of consecutive integers is an odd integer. [1]

Find the sum to n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots + \frac{2r+1}{r^2(r+1)^2} + \ldots$$

and deduce the sum to infinity of the series. [5]

9231/13/M/J/14 Q2

3. Summation of series

Q21.

1 Given that

$$u_k = \frac{1}{\sqrt{(2k-1)}} - \frac{1}{\sqrt{(2k+1)}},$$

express $\sum_{k=13}^{n} u_k$ in terms of n. [4]

Deduce the value of $\sum_{k=13}^{\infty} u_k$. [1]

9231/11/O/N/14 Q1

Q22.

1 Use the List of Formulae (MF10) to show that $\sum_{r=1}^{13}(3r^2 - 5r + 1)$ and $\sum_{r=0}^{9}(r^3 - 1)$ have the same numerical value. [4]

9231/11/M/J/15 Q1

Q23.

4 Use the formula for $\tan(A - B)$ in the List of Formulae (MF10) to show that

$$\tan^{-1}(x+1) - \tan^{-1}(x-1) = \tan^{-1}\left(\frac{2}{x^2}\right).$$ [3]

Deduce the sum to n terms of the series

$$\tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \ldots.$$ [4]

9231/13/M/J/15 Q4

Q24.

4 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2 - n + 1)}} - \frac{n+6}{\sqrt{(n^2 + n + 1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

9231/11/O/N/15 Q4

3. Summation of series

Q25.

2 Express $\frac{4}{r(r+1)(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{4}{r(r+1)(r+2)}$. [5]

Deduce the value of $\sum_{r=1}^{\infty} \frac{4}{r(r+1)(r+2)}$. [1]

9231/11/M/J/16 Q2

Q26.

1 Verify that $\frac{1}{(3r+1)(3r+4)} = \frac{1}{3}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$. [1]

Let S_N denote $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r+4)}$ and let S denote $\sum_{r=1}^{\infty} \frac{1}{(3r+1)(3r+4)}$. Find the least value of N such that $S - S_N < \frac{1}{10000}$. [5]

9231/13/M/J/16 Q1

Q27.

1 Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1}$. [4]

Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

9231/11/O/N/16 Q1

Q28.

1 It is given that $\sum_{r=1}^{n} u_r = n^2(2n+3)$, where n is a positive integer.

(i) Find $\sum_{r=n+1}^{2n} u_r$. [2]

(ii) Find u_r. [3]

9231/11/M/J/17 Q1

3. Summation of series

Q29.

2 **(i)** Verify that $\dfrac{2r+1}{r(r+1)(r+2)} = \dfrac{1}{2}\left\{\dfrac{(2r+1)(2r+3)}{(r+1)(r+2)} - \dfrac{(2r-1)(2r+1)}{r(r+1)}\right\}$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)} - \frac{3}{2}\right\}$. [2]

(iii) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{2r+1}{r(r+1)(r+2)}$. [2]

9231/13/M/J/17 Q2

Q30.

1 Find $\displaystyle\sum_{r=1}^{n}(4r-3)(4r+1)$, giving your answer in its simplest form. [4]

9231/11/O/N/17 Q1

Q31.

5 Let $S_n = \displaystyle\sum_{r=1}^{n}(-1)^{r-1}r^2$.

(i) Use the standard result for $\displaystyle\sum_{r=1}^{n} r^2$ given in the List of Formulae (MF10) to show that

$$S_{2n} = -n(2n+1).$$ [4]

(ii) State the value of $\displaystyle\lim_{n\to\infty}\frac{S_{2n}}{n^2}$ and find $\displaystyle\lim_{n\to\infty}\frac{S_{2n+1}}{n^2}$. [4]

9231/11/M/J/18 Q5

Q32.

2 **(i)** Verify that

$$\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}} = \frac{1}{n\mathrm{e}^{n}} - \frac{1}{(n+1)\mathrm{e}^{n+1}}.$$ [1]

Let $S_N = \displaystyle\sum_{n=1}^{N}\frac{n(\mathrm{e}-1)+\mathrm{e}}{n(n+1)\mathrm{e}^{n+1}}$.

(ii) Express S_N in terms of N and e. [2]

Let $S = \displaystyle\lim_{N\to\infty} S_N$.

(iii) Find the least value of N such that $(N+1)(S-S_N) < 10^{-3}$. [3]

9231/13/M/J/18 Q2

3. Summation of series

Q33.

(i) By considering $(2r+1)^2-(2r-1)^2$, use the method of differences to prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1).$$ [3]

(ii) By considering $(2r+1)^4-(2r-1)^4$, use the method of differences and the result given in part **(i)** to prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$ [5]

The sums S and T are defined as follows:

$$S = 1^3+2^3+3^3+4^3+\ldots+(2N)^3+(2N+1)^3,$$
$$T = 1^3+3^3+5^3+7^3+\ldots+(2N-1)^3+(2N+1)^3.$$

(iii) Use the result given in part **(ii)** to show that $S=(2N+1)^2(N+1)^2$. [1]

(iv) Hence, or otherwise, find an expression in terms of N for T, factorising your answer as far as possible. [2]

(v) Deduce the value of $\frac{S}{T}$ as $N\to\infty$. [2]

9231/11/O/N/18 Q11

Q34.

7 Let

$$S_N=\sum_{r=1}^{N}(3r+1)(3r+4) \quad \text{and} \quad T_N=\sum_{r=1}^{N}\frac{1}{(3r+1)(3r+4)}.$$

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = N(3N^2+12N+13).$$ [3]

(ii) Use the method of differences to show that

$$T_N=\frac{1}{12}-\frac{1}{3(3N+4)}.$$ [3]

(iii) Deduce that $\frac{S_N}{T_N}$ is an integer. [2]

(iv) Find $\lim_{N\to\infty}\frac{S_N}{N^3T_N}$. [2]

9231/12/O/N/18 Q7

3. Summation of series

Q35.

2 Let $u_n = \dfrac{4\sin(n-\frac{1}{2})\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}$.

(i) Using the formulae for $\cos P \pm \cos Q$ given in the List of Formulae MF10, show that

$$u_n = \frac{1}{\cos n} - \frac{1}{\cos(n-1)}. \qquad [2]$$

(ii) Use the method of differences to find $\sum_{n=1}^{N} u_n$. [2]

(iii) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

9231/11/M/J/19 Q2

Q36.

4 **(i)** Use the method of differences to show that $\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3} - \frac{1}{3(3N+1)}$. [4]

(ii) Find the limit, as $N \to \infty$, of $\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)}$. [4]

9231/13/M/J/19 Q4

Q37.

5 Let $S_N = \sum_{r=1}^{N}(5r+1)(5r+6)$ and $T_N = \sum_{r=1}^{N} \frac{1}{(5r+1)(5r+6)}$.

(i) Use standard results from the List of Formulae (MF10) to show that

$$S_N = \tfrac{1}{3}N(25N^2 + 90N + 83). \qquad [3]$$

(ii) Use the method of differences to express T_N in terms of N. [4]

(iii) Find $\lim_{N\to\infty}(N^{-3}S_N T_N)$. [2]

9231/11/O/N/19 Q5

3. Summation of series

Q38.

4 **(a)** By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

9231/11/M/J/20 Q4

Q39.

3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2-1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

9231/13/M/J/20 Q3

Q40.

2 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n} (7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{(7r+1)(7r+8)}$. [1]

9231/11/O/N/20 Q2

3. Summation of series

Q41.

3 **(a)** By simplifying $\left(x^n-\sqrt{x^{2n}+1}\right)\left(x^n+\sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n-\sqrt{x^{2n}+1}}=-x^n-\sqrt{x^{2n}+1}$. [1]

Let $u_n=x^{n+1}+\sqrt{x^{2n+2}+1}+\dfrac{1}{x^n-\sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N}u_n$ in terms of N and x. [3]

(c) Deduce the set of values of x for which the infinite series

$$u_1+u_2+u_3+\ldots$$

is convergent and give the sum to infinity when this exists. [3]

9231/12/O/N/20 Q3

Q42.

2 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n}\dfrac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty}\dfrac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$. [1]

9231/11/M/J/21 Q2

Q43.

1 **(a)** Show that

$$\tan(r+1)-\tan r=\frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

Let $u_r=\dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n}u_r$. [3]

(c) Explain why the infinite series $u_1+u_2+u_3+\ldots$ does not converge. [1]

9231/13/M/J/21 Q1

3. Summation of series

Q44.

(a) Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

9231/11/O/N/21 Q2

Q45.

Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

9231/12/O/N/21 Q3

Q46.

Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

9231/11/M/J/22 Q1

3. Summation of series

Q47.

Let $u_r = e^{rx}(e^{2x} - 2e^x + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

9231/13/M/J/22 Q4

Q48.

(a) By considering $(2r+1)^3 - (2r-1)^3$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$ [5]

Let $S_n = 1^2 + 3\times 2^2 + 3^2 + 3\times 4^2 + 5^2 + 3\times 6^2 + \ldots + \left(2 + (-1)^n\right)n^2$.

(b) Show that $S_{2n} = \tfrac{1}{3}n(2n+1)(an+b)$, where a and b are integers to be determined. [3]

(c) State the value of $\lim_{n\to\infty} \frac{S_{2n}}{n^3}$. [1]

9231/11/O/N/22 Q3

Q49.

(a) Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

9231/12/O/N/22 Q1

Solutions

Q1.

2 $u_n = \ln(1+x^{n+1}) - \ln(1+x)$ or for ln{Product of fractions} B1

$\sum_{n=1}^{N} u_n = S_N = \ln\left[\left(1+x^{N+1}\right)/(1+x)\right]$ (AEF) Cancels → result M1A1

(i) $S_\infty = -\ln(1+x)$ **OR** $\ln\left(\frac{1}{1+x}\right)$ A1

(ii) $S_\infty = 0$ B1

Q2.

9 Set up

$$H_k : \sum_{n=1}^{k} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)}$$ B1

for some positive integer k

$$H_k \Rightarrow \sum_{n=1}^{k+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)} + \frac{4k+5}{(k+1)(k+2)(2k+1)(2k+3)}$$ M1

$$= 1 - \frac{2k^2+3k+1}{(k+1)(k+2)(2k+1)(2k+3)}$$ A1

$$= \ldots = 1 - \frac{1}{(k+2)(2k+3)}$$ A1

Verifies H_1 is true. B1

Correct completion of induction argument A1

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = \ldots = \frac{1}{(N+1)(2N+1)} - \frac{1}{(2N+1)(4N+1)}$$ M1A1

$$= \frac{3N}{(N+1)(2N+1)(4N+1)} < \frac{3N}{N.2N.4N} = \frac{3}{8N^2}$$ M1A1

OR

$$= \frac{3N}{8N^3+14N^2+7N+1} = \frac{3}{8N^2+14N+7+\frac{1}{N}}$$

Since $N \geqslant 1$ $14N + 7 + \frac{1}{N} > 0$

$$\therefore \sum < \frac{3}{8N^2}$$

3. Summation of series

Q3.

2 Verifies displayed result — M1A1

(i) $S_N = 1/15 - 1/(N+3)(2N+5)$ — M1A1A1

(ii) $S_\infty = 1/15$ — A1 ft

Note: Must see working for preliminary result

Either $(2n^2 + 11n + 15) - (2n^2 + 7n + 6)$ (oe)

Or $(n+3)(2n+5) - (n+2)(2n+3)$

} in numerator

A1 A1

Q4.

11 EITHER

$H_k : S_k = \sum_{n=1}^{k} n^3 = (1/4)k^2(k+1)^2$ for some k — B1

$H_k \Rightarrow S_{k+1} = (1/4)k^2(k+1)^2 + (k+1)^3$ — M1

$= \ldots = (1/4)(k+1)^2(k+2)^2$ so that $H_k \Rightarrow H_{k+1}$ — M1A1

Verifies H_1 is true and completes induction argument — A1

$\sum_{n=1}^{N}(20n^3 + 36n^2) = 5N^2(N+1)^2 + 6N(N+1)(2N+1)$ — M1

$= \ldots = N(N+1)(N+3)(5N+2)$ (AG) — M1A1

$S_N = N(N+1)(N+3)(5N+2) + (\mu/2)N(N+1)$ — M1

$= N(N+1)(5N^2 + 17N + 6 + \mu/2)$ — M1

Take $\mu = -12$, then $S_N = N^2(N+1)(5N+17)$ so that $a = 5,\ b = 17$ — A1

$N^{-4}S_N = 5 + 22/N + 17/N^2,\ > 5 + 22/N,\ \forall N \geqslant 1$ — M1, A1

$N \geqslant 18 \Rightarrow N > 17 \Rightarrow 17/N^2 < 1/N$

$\Rightarrow N^{-4}S_N < 5 + 23/N$ (AG) — A1

Q5.

4 $(N + 1/2)^6 - 1/64 = 6S_N + (5/4)N^2(N+1)^2 + 3N(N+1)/16$ — M1A1A1

M1 for application of difference method:

A1 for LHS correct: A1 for RHS correct

$S_N = (1/6)(N + 1/2)^6 - (5/24)N^2(N+1)^2 - (1/32)N(N+1) - 1/384$

Or $\frac{1}{6}\left\{(N+\frac{1}{2})^6 - (\frac{1}{2})^6 - \frac{5N^2(N+1)^2}{4} - \frac{3}{16}N(N+1)\right\}$ — A1

[4]

(i) For $\lambda = 6$, $S_\infty = 1/6$ — B2

(ii) For $\lambda > 6$, $S_\infty = 0$ — B1

[3]

3. Summation of series

Q6.

2 $2\sin\alpha\sum_{n=1}^{N}\sin(2n\alpha)=\cos\alpha-\cos[(2N+1)\alpha]$ M1A1

$\Rightarrow$ displayed result (AG) M1A1 [4]

$\cos(2N+1)\pi/3$ oscillates finitely as $n\to\infty\Rightarrow\sum_{n=1}^{\infty}\sin(2n\pi/3)$ does not converge (CWO) B1

Require $\alpha=\frac{\pi}{3}$, 'oscillate' or values of $\cos(2N+1)\frac{\pi}{3}$ given as $\frac{1}{2}$ or –1 [1]

Q7.

2 nth term is $\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$ M1A1

$$S_N=\frac{1}{2}\left[\begin{array}{l}\left(\frac{1}{N}-\frac{1}{N+2}\right)+\left(\frac{1}{N-1}-\frac{1}{N+1}\right)+\left(\frac{1}{N-2}-\frac{1}{N}\right)+\ldots\\ \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{1}-\frac{1}{3}\right)\end{array}\right]$$ M1 sum of terms

$=\frac{1}{2}\left[\frac{3}{2}-\frac{1}{N+2}-\frac{1}{N+1}\right]$ A1 after cancellation [4]

Limit = ¾ B1√ [1]

Q8.

1	Any method including cover-up rule.	$\frac{1}{(2r+1)(2r+3)}=\frac{1}{2}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$	B1		
	Expresses all terms as differences.	$S_n=\frac{1}{2}\left(\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\ldots\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right)$	M1A1		
	Finds sum.	$=\frac{1}{6}-\frac{1}{2(2n+3)}$ (acf)	A1	4	
		$S_\infty=\frac{1}{6}$ (B0M1A1√ A0A1√ if signs reversed.)	A1	1	**[5]**

3. Summation of series

Q9.

1	Finds four times sum of first n squares. Subtracts eight times sum of first n squares from sum of first $2n$ squares. Simplifies.	$2^2+4^2+...+(2n)^2=\frac{4n(n+1)(2n+1)}{6}$ $1^2-2^2+3^2-4^2+...-(2n)^2$ $=\frac{2n(2n+1)(4n+1)}{6}-\frac{8n(n+1)(2n+1)}{6}$ $=\frac{n(2n+1)}{3}(4n+1-4n-4)=-n(2n+1)$ Or $\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n-\frac{4n(n+1)(2n+1)}{6}$ $=-2n^2-n$	M1A1 M1A1 A1 (M1A1) (A1)	2 3	**[5]**

Q10.

1	Verifies result.	$\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{n^2+2n+1-n^2}{n^2(n+1)^2}=\frac{2n+1}{n^2(n+1)^2}$ (AG)	B1	1	
	Uses difference method	$S_N=\left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+.....+\left(\frac{1}{N^2}-\frac{1}{(N+1)^2}\right)$	M1		
	to sum.	$=1-\frac{1}{(N+1)^2}$	A1	2	
	Considers difference between sum and sum to infinity.	$S-S_N<10^{-16}\Rightarrow\frac{1}{(N+1)^2}<10^{-16}$	M1		
		$\Rightarrow(N+1)>10^8$	A1		
	Solves inequality.	$\Rightarrow$ *least* $N=10^8$	A1	3	**[6]**

3. Summation of series

Q11.

3	Proves initial result.	$f(r-1)-f(r)=\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}$	M1		
		$=\frac{r+2-r}{r(r+1)(r+2)}=\frac{2}{r(r+1)(r+2)}$ (AG)	A1	2	
	Sets up method of differences.	$\sum_1^n \frac{1}{r(r+1)(r+2)}=\frac{1}{2}\left\{\frac{1}{1\times 2}-\frac{1}{2\times 3}\right\}\ldots\ldots$	M1A1		
		$+\frac{1}{2}\left\{\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right\}$			
	Shows cancellation to get result.	$=\frac{1}{4}-\frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	A1	3	
	States sum to infinity.	$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)}=\frac{1}{4}$	A1√	1	**[6]**
	'Non hence' method for last two parts	$\frac{1}{r(r+1)(r-2)}=\frac{1}{2r}-\frac{1}{(r+1)}+\frac{1}{2(r+2)}$	(M1)		
	i.e. penalty of 1 mark.	$\Rightarrow\cdots\Rightarrow$			
		$\frac{1}{2}-\frac{1}{2}+\frac{1}{4}\ldots+\frac{1}{2(n+1)}-\frac{1}{(n+1)}+\frac{1}{2(n+2)}$	(A1)		
		$=\frac{1}{4}-\frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	(A1)	(3)	
		$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)}=\frac{1}{4}$	(A1√)	(1)	

Q12.

1	Finds partial fractions.	$\frac{1}{r(r+2)}=\frac{1}{2}\left\{\frac{1}{r}-\frac{1}{r+2}\right\}$	M1A1		
		$\sum_{r=1}^n \frac{1}{r(r+2)}=$			
	Use method of differences.	$\frac{1}{2}\left\{\left[\frac{1}{n}-\frac{1}{n+2}\right]+\left[\frac{1}{n-1}-\frac{1}{n+1}\right]+\ldots+\left[\frac{1}{2}-\frac{1}{4}\right]+\left[1-\frac{1}{3}\right]\right\}$	M1		
	Obtains results.	$=\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}$ (acf) $\Rightarrow S_\infty=\frac{3}{4}$	A1A1√	5	**[5]**

3. Summation of series

Q13.

4	Verifies given result.	$r(r+1)(r+2)-(r-1)r(r+1)=r(r+1)(r+2-r+1)$ $=3r(r+1)$ (AG)	B1	1	
	Uses method of differences to sum first series.	$\sum_{r=1}^{n} r(r+1)=$ $\frac{1}{3}\{[f(n)-f(n-1)]+[f(n-1)-f(n-2)]+...+[f(1)-f(0)]\}$	M1		
		$=\frac{1}{3}n(n+1)(n+2)$ (AG) (Award B1 if 'not hence'.)	A1	2	
	Subtracts $\sum_{r=1}^{n} r$ to obtain sum of second series.	$\sum_{r=1}^{n} r^2=\sum_{r=1}^{n} r(r+1)-\sum_{r=1}^{n} r=\frac{n(n+1)(n+2)}{3}-\frac{n(n+1)}{2}$	M1		
		$=\frac{1}{6}n(n+1)(2n+4-3)=\frac{1}{6}n(n+1)(2n+1)$ (AG)	A1	2	
	Splits series into two series.	$(1^2+2^2+...+n^2)+(2^2+4^2+...+(n-1)^2)=$	M1		
	Applies sum of squares formula to obtain result.	$\frac{n(n+1)(2n+1)}{6}+\frac{4\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)n}{6}=...=\frac{1}{2}n^2(n+1)$	M1A1	3	**[8]**

Q14.

1	Use of:	$\sum_{n+1}^{2n}=\sum_{1}^{2n}-\sum_{1}^{n}$	M1		
	Use of:	$\sum_{1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$	M1		
	Obtains result.	$\frac{2n(2n+1)(4n+1)}{6}-\frac{n(n+1)(2n+1)}{6}$	A1		
		$=\frac{1}{6}n(2n+1)(8n+2-n-1)=\frac{1}{6}n(2n+1)(7n+1)$ (AG)	A1	4	**[4]**

3. Summation of series

Q15.

5	Finds partial fractions.	$\frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\left\{\frac{1}{2r+1} - \frac{1}{2r+3}\right\}$ $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$	M1A1		
	Expresses terms as differences.	$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2N+1} - \frac{1}{2N+3}\right)$	M1A1		
	Shows cancellation.	$= \frac{1}{6} - \frac{1}{2(2N+3)}$ (AG)	A1	5	
	Uses $\sum_{N+1}^{2N} = \sum_{1}^{2N} - \sum_{1}^{N}$.	$\sum_{N+1}^{2N} = \left(\frac{1}{6} - \frac{1}{2(4N+3)}\right) - \left(\frac{1}{6} - \frac{1}{2(2N+3)}\right)$	M1		
	Applies result	$= \frac{1}{2}\left(\frac{1}{2N+3} - \frac{1}{4N+3}\right)$	A1		
	and simplifies.	$= \frac{N}{(2N+3)(4N+3)}$	M1		
	Deduces inequality.	$< \frac{N}{2N.4N} = \frac{1}{8N}$ (AG)	A1	4	**[9]**

Q16.

1	Simplifies.	$f(r+1) - f(r) = r(r+1)!-(r-1)r!$ $= r!(r^2 + r - r + 1) = r!(r^2 + 1)$	M1 A1		
	Uses difference method.	$\sum_{1}^{n} = f(2) - f(1) + f(3) - f(2) + \ldots f(n+1) - f(n)$ $= n(n+1)!-0 = n(n+1)!$	M1 A1		
	Obtains result.	$\therefore \sum_{n+1}^{2n} = 2n(2n+1)!-n(n+1)!$ (Or directly using $\sum_{n+1}^{2n} = f(2n+1) - f(n+1)$ from the method of differences.)	A1	5	**[5]**

3. Summation of series

Q17.

3	Writes first four sums.	$S_1 \ldots S_4 \sim 3, 10, 21, 36$	B1		
	Deduces first four terms, conjectures and justifies result.	$u_1 \ldots u_4 \sim 3, 7, 11, 15 \Rightarrow u_r = 4r - 1$	B1B1		
		since $S_n = \frac{n}{2}\{6 + 4(n-1)\} = 2n^2 + n$ as given.	B1		
		Or $u_r = S_r - S_{r-1} = 2r^2 + r - 2(r-1)^2 - (r-1)$	B1B1		
		$= 4r - 1$	B1	4	
	Obtains required sum.	$\Sigma_{n+1}^{2n}(4r-1) = 4.\frac{2n(2n+1)}{2} - 2n - \left(4.\frac{n(n+1)}{2} - n\right)$	M1A1		
		$= 8n^2 + 2n - (2n^2 + n) = 6n^2 + n$	A1	3	
		Or Sum of AP $= \frac{n}{2}(4n + 3 + 8n - 1) = 6n^2 + n$			**[7]**

Q18.

1	Finds partial fractions.	$\frac{1}{r(r-1)(r+1)} = \frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}$	B1	**(1)**	
	Expresses each term in fractions	$\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) \ldots \left(\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$	M1A1		
	Cancels terms and sums	$= \frac{1}{4} - \frac{1}{2n} + \frac{1}{2(n+1)}$ (OE)	M1A1	**(4)**	
	Find sums to infinity	$S\infty = \frac{1}{4}$	B1	**(1)**	**[6]**

Q19.

2	$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$	B1 [1]
	$(n+1)^4 - 1^4 = 4\Sigma_{r=1}^{n} r^3 + 6\Sigma_{r=1}^{n} r^2 + 4\Sigma_{r=1}^{n} r + n$	M1
	$n^4 + 4n^3 + 6n^2 + 4n = 4\Sigma_{r=1}^{n} r^3 + n(2n^2 + 3n + 1) + 2n^2 + 2n + n$	A1A1
	$\Rightarrow \ldots \Rightarrow \Sigma_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$. (AG)	A1 [4]

3. Summation of series

Q20.

2	$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \Rightarrow$ odd. $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \ldots \frac{2n+1}{n^2(n+1)^2} = \frac{2^2-1^2}{1^2.2^2} + \frac{3^2-2^2}{2^2.3^2} + \frac{4^2-3^2}{3^2.4^2} + \ldots \frac{(n+1)^2-n^2}{n^2(n+1)^2}$ $= 1 - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots + \frac{1}{n^2} - \frac{1}{(n+1)^2}$ $= 1 - \frac{1}{(n+1)^2}$ Sum to infinity $=1$.	B1 [1] M1A1 M1 A1 A1√ [5]

Q21.

1	$\left(\frac{1}{\sqrt{25}} - \frac{1}{\sqrt{27}}\right) + \left(\frac{1}{\sqrt{27}} - \frac{1}{\sqrt{29}}\right) + \ldots + \left(\frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n+1}}\right)$ $\sum_{r=13}^{n} u_k = \frac{1}{5} - \frac{1}{\sqrt{2n+1}}$ $\sum_{r=13}^{\infty} u_k = \frac{1}{5}$	M1A1 M1A1 (4) B1√ (1) **[5]**

Q22.

1	$3 \times \frac{13 \times 14 \times 27}{6} - 5 \times \frac{13 \times 14}{2} + 13 = 2015$ $\left[\frac{9 \times 10}{2}\right]^2 - 10 = 2015$ (Award M1 for subtracting 9 or 10 here.)	M1A1 M1A1 (4) **Total: 4**

Q23.

4	$A = \tan^{-1}(x+1)$; $B = \tan^{-1}(x-1) \Rightarrow \tan(A-B) = \frac{(x+1)-(x-1)}{1+(x+1)(x-1)}$ $\tan(A-B) = \frac{2}{x^2} \Rightarrow A - B = \tan^{-1}\left(\frac{2}{x^2}\right)$ (AG) LHS $= (\tan^{-1}2 - \tan^{-1}0) + (\tan^{-1}3 - \tan^{-1}1) + \ldots + (\tan^{-1}n - \tan^{-1}[n-2])$ $+ (\tan^{-1}[n+1] - \tan^{-1}[n-1])$. $= \tan^{-1}[n+1] + \tan^{-1}n - \tan^{-1}1 - \tan^{-1}0 = \tan^{-1}[n+1] + \tan^{-1}n - \frac{1}{4}\pi$	M1A1 A1 (3) M1A1 A1A1 (4) **Total 7**

3. Summation of series

Q24.

4	(i)	$\left(\frac{6}{\sqrt{1}}-\frac{7}{\sqrt{3}}\right)+\left(\frac{7}{\sqrt{3}}-\frac{8}{\sqrt{7}}\right)+\ldots+\left(\frac{35}{\sqrt{871}}-\frac{36}{\sqrt{931}}\right)=6-\frac{36}{\sqrt{931}}=4.820$	**M1A1** **A1** [3]
	(ii)	$6-\frac{n+6}{\sqrt{n^2+n+1}}>4.9\Rightarrow 0.21n^2-10.79n-34.79(>0)$ $\Rightarrow n>54.42\ldots$ so 55 terms required.	**M1A1** **dM1A1** [4] **Total 7**

Q25.

2	$\frac{2}{r}-\frac{4}{r+1}+\frac{2}{r+2}$ (Award **B2** if written down by cover up rule.)	**M1A1**
	$\left(2-2+\frac{2}{3}\right)+\left(1-\frac{4}{3}+\frac{1}{2}\right)+\ldots+\left(\frac{2}{n-1}-\frac{4}{n}+\frac{2}{n+1}\right)+\left(\frac{2}{n}-\frac{4}{n+1}+\frac{2}{n+2}\right)$	**M1A1**
	$=1-\frac{2}{n+1}+\frac{2}{n+2}$ (AEF)	**A1** [5]
	Sum to infinity = 1	**B1**√ [1]

Q26.

1	$\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)=\frac{1}{3}\left(\frac{(3r+4)-(3r+1)}{(3r+1)(3r+4)}\right)=\frac{1}{(3r+1)(3r+4)}$ **AG**	**B1** [1]
	$S_N=\frac{1}{3}\left[\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots+\left(\left(\frac{1}{3N+1}\right)-\frac{1}{3N+4}\right)\right]=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)$	**M1 A1**
	$\Rightarrow S=\frac{1}{12}$	**A1**
	$\Rightarrow S-S_N=\frac{1}{3(3N+4)}<\frac{1}{10000}$	**M1**
	$\Rightarrow 3N+4>\frac{10000}{3}\Rightarrow N>1109\frac{7}{9}$. Thus least N is 1110.	**A1** [5]

3. Summation of series

Q27.

1	$\frac{1}{(2r-1)(2r+1)}=\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ $\sum_{r=1}^{n}\frac{1}{(2r)^2-1}=\frac{1}{2}\left(\left[1-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{5}\right]+\ldots+\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]\right)=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$ (OE) $\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}\Rightarrow\sum_{r=1}^{\infty}\frac{1}{(2r)^2-1}=\frac{1}{2}$	M1A1 M1A1 B1	[4] [1]

Q28.

1(i)	$\sum_{n+1}^{2n}u_r=(2n)^2(4n+3)-n^2(2n+3)$	M1	Method mark for using $S_{2n}-S_n$
	$=14n^3+9n^2$	A1	
	Total:	2	
1(ii)	$u_r=r^2(2r+3)-(r-1)^2(2r+1)$	M1A1	Method mark for using S_r-S_{r-1} OE
	$=6r^2-1$	A1	SR: CAO **B1** without wrong working
	Total:	3	

Q29.

2(i)	$\text{RHS}=\frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^2-1)(r+2)}{r(r+1)(r+2)}\right\}$	M1	
	$=\frac{1}{2}\left\{\frac{4r^3+8r^2+3r-(4r^3+8r^2-r-2)}{r(r+1)(r+2)}\right\}=\frac{1}{2}\left\{\frac{(4r+2)}{r(r+1)(r+2)}\right\}=\frac{(2r+1)}{r(r+1)(r+2)}$	A1	AG
	Total:	2	
2(ii)	Sum to n terms is: $\frac{1}{2}\left\{\left[\frac{3.5}{2.3}-\frac{1.3}{1.2}\right]+\left[\frac{5.7}{3.4}-\frac{3.5}{2.3}\right]+\ldots+\left[\frac{(2n-1)(2n+1)}{n(n+1)}-\frac{(2n-3)(2n-1)}{n(n-1)}\right]+\left[\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{(2n-1)(2n+1)}{n(n+1)}\right]\right\}$	M1	
	$=\frac{1}{2}\left\{\frac{(2n+1)(2n+3)}{(n+1)(n+2)}-\frac{3}{2}\right\}$	A1	AG
2(iii)	$S_\infty=\frac{1}{2}\times4-\frac{3}{4}=1\frac{1}{4}$.	M1A1	
	Total:	4	

3. Summation of series

Q30.

1	$\sum_{r=1}^{n} u_r = 16\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r - 3n$	**M1A1**	**M1** for split into 3 parts
	$= 16\frac{n(n+1)(2n+1)}{6} - 8\frac{n(n+1)}{2} - 3n$	**M1**	For using formulae correctly in their expression
	$= \ldots = \frac{n}{3}\left(16n^2 + 12n - 13\right)$ (3 terms)	**A1**	OE
		4	

Q31.

5(i)	$S_{2n} = 1^2 - 2^2 + 3^2 - 4^2 \ldots\ldots$	**M1**	Uses correct difference.
	so $S_{2n} = \sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n} (2r)^2 = \sum_{r=1}^{2n} r^2 - 8\sum_{r=1}^{n} r^2$	**A1**	Alt method: Use $\sum_1^n (2r-1)^2 - \sum_1^n (2r)^2$ =A1
	Thus $S_{2n} = \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{8}{6}n(n+1)(2n+1)$	**M1**	$\sum_1^n 4r^2 - 4\sum_1^n (r) + n - 4\sum_1^n (r)^2$ M1
	Factorising, $S_{2n} = \frac{1}{3}n(2n+1)(4n+1-4n-4) = -n(2n+1)$	**A1**	$= -n(2n+1)$ A1 AG
		4	
5(ii)	$\lim_{n\to\infty} \frac{S_{2n}}{n^2} = -2$	**B1**	
	$S_{2n+1} = S_{2n} + (-1)^{2n}(2n+1)^2$	**M1**	
	So, $S_{2n+1} = -n(2n+1) + (2n+1)^2 = (2n+1)(n+1)$	**M1**	Uses the result given in **(i)** or using $\lim_{n\to\infty} \frac{S_{2n}}{n^2}$ and correct sign
	Thus $\lim_{n\to\infty} \frac{S_{2n+1}}{n^2} = 2$	**A1**	Alt: Find limit from previous line directly
		4	

Q32.

2(i)	$\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}} = \frac{(n+1)e - n}{n(n+1)e^{n+1}} = \frac{n(e-1)+e}{n(n+1)e^{n+1}}$	**B1**	Verifies result (AG).
2(ii)	$S_N = \sum_{n=1}^{N}\left(\frac{1}{ne^n} - \frac{1}{(n+1)e^{n+1}}\right) = (\frac{1}{e} - \cancel{\frac{1}{2e^2}} + \cancel{\frac{1}{2e^2}} - \cancel{\frac{1}{3e^3}} + \ldots\ldots\ldots\ldots \cancel{\frac{1}{Ne^N}} - \frac{1}{(N+1)e^{N+1}})$ SOI =	**M1**	Uses difference method to sum.
	$\frac{1}{e} - \frac{1}{(N+1)e^{N+1}}$.	**A1**	
2(iii)	$S = \frac{1}{e}$	**B1**	Finds S
	$(N+1)(S - S_N) < 10^{-3} \Rightarrow \frac{1}{e^{N+1}} < 10^{-3}$	**M1**	Attempts to find difference between sum and sum to infinity.
	$\Rightarrow e^{N+1} > 10^3$ $\Rightarrow$ least such N is 6.	**A1**	
		6	

3. Summation of series

Q33.

11E(i)	$(2r+1)^2-(2r-1)^2=8r$	**B1**	
	$\Rightarrow 8\sum_{r=1}^{n} r=-1^2+(2n+1)^2$	**M1**	Sums both sides and uses method of differences.
	$\Rightarrow \sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$	**A1**	AG.
		3	
11E(ii)	$(2r+1)^4-(2r-1)^4=\left((2r+1)^2+(2r-1)^2\right)\left((2r+1)^2-(2r-1)^2\right)$	**M1**	Uses difference of squares or expands.
	$=\left(8r^2+2\right)(8r)=16\left(4r^3+r\right)$	**A1**	
	$\Rightarrow 4\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=-\left(1-\frac{1}{2}\right)^4+\left(n+\frac{1}{2}\right)^4$	**M1**	Sums both sides and uses method of differences.
	$4\sum_{r=1}^{n} r^3=\left(n+\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^4-\frac{1}{2}n(n+1)$	**M1**	Uses formula for $\sum r$
	$=\left(\left(n+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)\left(\left(n+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)-\frac{1}{2}n(n+1)=n^2(n+1)^2$	**A1**	AG
		5	
11E(iii)	$S=\frac{1}{4}(2N+1)^2(2N+2)^2=(2N+1)^2(N+1)^2$	**B1**	Uses formula for $\sum r^3$. AG.
		1	
11E(iv)	$T=S-\sum_{r=1}^{N}(2r)^3=(2N+1)^2(N+1)^2-2N^2(N+1)^2$	**M1**	Eliminates even terms from S.
	$(N+1)^2\left(2N^2+4N+1\right)$	**A1**	Accept $(N+1)^2(2N(N+2)+1)$.
		2	
11E(v)	$\frac{S}{T}=\frac{(2N+1)^2}{2N^2+4N+1}=\frac{4N^2+4N+1}{2N^2+4N+1}$	**M1**	Writes fraction as quadratic in N divided by quadratic in N.
	Converges to 2 as $N\to\infty$.	**A1**	
		2	
11O(i)	$x=-1\Rightarrow y^3+2y-3=0\Rightarrow(y-1)\left(y^2+y+3\right)=0$	**M1**	Considers cubic polynomial in y.
	There is only one real root $(=1)$.	**A1**	
		2	

3. Summation of series

Q34.

7(i)	$\sum_{r=1}^{N}(3r+1)(3r+4)=9\sum_{r=1}^{N}r^2+15\sum_{r=1}^{N}r+4N$	**M1**	Expands
	$9\left(\frac{1}{6}N(N+1)(2N+1)\right)+15\left(\frac{1}{2}N(N+1)\right)+4N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{9}{6}(2N^2+3N+1)+\frac{15}{2}N+\frac{15}{2}+4\right)$ $=N(3N^2+12N+13)$	**A1**	Shows simplification to the given answer (AG).
		3	
7(ii)	$\frac{1}{(3r+1)(3r+4)}=\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)$	**B1**	Finds partial fractions.
	$T_N=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3(N-1)+1}-\frac{1}{3N+4}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)=\frac{1}{12}-\frac{1}{3(3N+4)}$	**A1**	Cancels to given answer (AG).
		3	
7(iii)	$T_N=\frac{N}{4(3N+4)}\Rightarrow \frac{S_N}{T_N}=4(3N+4)(3N^2+12N+13)$	**M1**	Writes $\frac{S_N}{T_N}$ as a polynomial
	So $\frac{S_N}{T_N}$ is an integer because all terms are integers	**A1**	Justifies expression being integer
		2	
7(iv)	$\frac{S_N}{N^3T_N}=\frac{4(3N+4)(3N^2+16N+9)}{N^3}$	**M1**	Divides expression in **(iii)** by N^3 and takes limit
	$\rightarrow 4(3)(3)=36$	**A1**	
		2	

Q35.

2(i)	$\frac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\frac{2(\cos(n-1)-\cos n)}{2\cos n\cos(n-1)}$	**M1**	Uses formulae for $\cos P \pm \cos Q$.
	$\frac{1}{\cos n}-\frac{1}{\cos(n-1)}$	**A1**	AG
		2	
2(ii)	$\sum_{n=1}^{N}\frac{4\sin\left(n-\frac{1}{2}\right)\sin\frac{1}{2}}{\cos(2n-1)+\cos 1}=\sum_{n=1}^{N}\frac{1}{\cos n}-\frac{1}{\cos(n-1)}$ $=\frac{1}{\cos 1}-\frac{1}{1}+\frac{1}{\cos 2}-\frac{1}{\cos 1}+\ldots+\frac{1}{\cos N}-\frac{1}{\cos(N-1)}$	**M1**	Applies (i), shows enough terms and cancelation.
	$-1+\frac{1}{\cos N}$	**A1**	
		2	
2(iii)	$\cos N$ oscillates as $N\rightarrow\infty$ so $u_1+u_2+u_3+\ldots$ does not converge.	**B1**	States "oscillates" or refers to diverging values of $\cos N$.
		1	

3. Summation of series

Q36.

4(i)	$\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{3r-2}-\frac{1}{3r+1}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{N}\frac{1}{(3r+1)(3r-2)}=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\ldots\frac{1}{3N-2}-\frac{1}{3N+1}\right)$	**M1**	At least 3 term including final term.
	$=\frac{1}{3}\left(1-\frac{1}{3N+1}\right)$	**A1**	AG
		4	
4(ii)	$\sum_{r=N+1}^{N^2}\frac{N}{(3r+1)(3r-2)}=\sum_{r=1}^{N^2}\frac{N}{(3r+1)(3r-2)}-\sum_{r=1}^{N}\frac{N}{(3r+1)(3r-2)}$	**M1**	Uses $\sum_{r=N+1}^{N^2}=\sum_{r=1}^{N^2}-\sum_{r=1}^{N}$
	$=\frac{N}{3}-\frac{N}{3\left(3N^2+1\right)}-\left(\frac{N}{3}-\frac{N}{3(3N+1)}\right)$	**M1**	Applies **(i)**
	$=\frac{N}{3(3N+1)}-\frac{N}{3\left(3N^2+1\right)}=\frac{N^3-N^2}{(3N+1)\left(3N^2+1\right)}$	**A1**	Allow simplification to common denominator.
	$\to\frac{1}{9}$ as $N\to\infty$	**B1**	
		4	

Q37.

5(i)	$\sum_{r=1}^{N}(5r+1)(5r+6)=25\sum_{r=1}^{N}r^2+35\sum_{r=1}^{N}r+6N$	**M1**	Expands.
	$25\left(\frac{1}{6}N(N+1)(2N+1)\right)+35\left(\frac{1}{2}N(N+1)\right)+6N$	**M1**	Substitutes formulae for $\sum r$ and $\sum r^2$.
	$=N\left(\frac{25}{6}\left(2N^2+3N+1\right)+\frac{35}{2}N+\frac{35}{2}+6\right)=\frac{1}{3}N\left(25N^2+90N+83\right)$	**A1**	Simplifies to the given answer (AG).
		3	
5(ii)	$\frac{1}{(5r+1)(5r+6)}=\frac{1}{5}\left(\frac{1}{5r+1}-\frac{1}{5r+6}\right)$	**M1 A1**	Finds partial fractions.
	$T_N=\frac{1}{5}\left(\frac{1}{6}-\frac{1}{11}+\frac{1}{11}-\frac{1}{16}+\cdots+\frac{1}{5N+1}-\frac{1}{5N+6}\right)$	**M1**	Expresses terms as differences.
	$\frac{1}{5}\left(\frac{1}{6}-\frac{1}{5N+6}\right)=\frac{1}{30}-\frac{1}{5(5N+6)}$	**A1**	At least 3 terms including last.
		4	
5(iii)	$\frac{S_N}{N^3}T_N\to\frac{25}{3}\times\frac{1}{30}=\frac{5}{18}$	**M1 A1**	Divides S_N by N^3 and takes limits as $N\to\infty$
		2	

3. Summation of series

Q38.

4(a)	$\frac{1}{r^2-1}=\frac{1}{(r-1)(r+1)}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$	**M1 A1**
	$\sum_{r=2}^{n}\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\cdots+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}\right)$	**M1**
	$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$	**A1**
	$=\frac{3}{4}-\frac{2n+1}{2n(n+1)}$	**A1**
		5
4(b)	$\frac{3}{4}$	**B1**
		1
4(c)	$\sum_{r=n+1}^{2n}\frac{n}{r^2-1}=\sum_{r=2}^{2n}\frac{n}{r^2-1}-\sum_{r=2}^{n}\frac{n}{r^2-1}$	**M1**
	$=n\left(\frac{3}{4}-\frac{4n+1}{4n(2n+1)}\right)-n\left(\frac{3}{4}-\frac{2n+1}{2n(n+1)}\right)=\frac{n(2n+1)}{2n(n+1)}-\frac{n(4n+1)}{4n(2n+1)}$	**M1A1**
	$\to\frac{1}{2}$ as $n\to\infty$	**A1**
		4

Q39.

3(a)	$S_n=\sum_{r=1}^{n}(4r-2)^2=16\sum_{r=1}^{n}r^2-16\sum_{r=1}^{n}r+4n$	**M1 A1**
	$=\frac{8}{3}n(n+1)(2n+1)-8n(n+1)+4n$	**M1**
	$\frac{4}{3}n(4n^2+6n+2-6n-6+3)=\frac{4}{3}n\left(4n^2-1\right)$ **AG**	**A1**
		4
3(b)	$\frac{n}{S_n}=\frac{3}{4(2n-1)(2n+1)}=\frac{3}{8}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$	**M1 A1**
	$\sum_{n=1}^{N}\frac{n}{S_n}=\frac{3}{8}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}\ldots+\frac{1}{2N-1}-\frac{1}{2N+1}\right)$	**M1**
	$=\frac{3}{8}\left(1-\frac{1}{2N+1}\right)$	**A1**
		4
3(c)	$\frac{3}{8}$	**B1**
		1

3. Summation of series

Q40.

2(a)	$\sum_{r=1}^{n}(7r+1)(7r+8)=\sum_{r=1}^{n}49r^2+63r+8$	**M1**	Expands.
	$=49\left(\frac{1}{6}n(n+1)(2n+1)\right)+63\left(\frac{1}{2}n(n+1)\right)+8n$	**M1**	Substitutes formulae for $\sum r^2$ and $\sum r$.
	$=\frac{49}{3}n^3+56n^2+\frac{143}{3}n$	**A1**	
		3	
2(b)	$\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{7r+1}-\frac{1}{7r+8}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{(7r+1)(7r+8)}=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{15}+\frac{1}{15}-\frac{1}{22}+\ldots+\frac{1}{7n+1}-\frac{1}{7n+8}\right)$	**M1**	Writes at least three correct terms, including first and last.
	$=\frac{1}{7}\left(\frac{1}{8}-\frac{1}{7n+8}\right)$	**A1**	
		4	
2(c)	$\frac{1}{56}$	**B1 FT**	
		1	

Q41.

3(a)	$\left(x^n-\sqrt{\left(x^{2n}+1\right)}\right)\left(x^n+\sqrt{\left(x^{2n}+1\right)}\right)=x^{2n}-\left(x^{2n}+1\right)=-1$	**B1**	Uses the difference of two squares and obtains the given answer. AG
		1	
3(b)	$u_n=x^{n+1}+\sqrt{\left(x^{2n+2}+1\right)}-x^n-\sqrt{\left(x^{2n}+1\right)}$	**B1**	Uses the identity given in part **(a)**.
	$\sum_{n=1}^{N}u_n=x^2+\sqrt{\left(x^4+1\right)}-x-\sqrt{\left(x^2+1\right)}+x^3+\sqrt{\left(x^6+1\right)}-x^2-\sqrt{\left(x^4+1\right)}$ $+\ldots+x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x^N-\sqrt{\left(x^{2N}+1\right)}$	**M1**	Writes at least three complete correct terms, including the first and last.
	$=x^{N+1}+\sqrt{\left(x^{2N+2}+1\right)}-x-\sqrt{\left(x^2+1\right)}$	**A1**	(Don't allow in terms of n.)
		3	
3(c)	$-1<x<1$	**B1**	
	$x^{N+1}\to 0$ as $n\to\infty\Rightarrow u_1+u_2+u_3+\ldots=1-x-\sqrt{\left(x^2+1\right)}$	**M1 A1**	Finds sum to infinity.
		3	

3. Summation of series

Q42.

2(a)	$n-\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)(2n+1)$	**M1A1**	Substitutes correct formulae from MF19. Expanding brackets correctly.
	$\frac{1}{3}n-n^2-\frac{1}{3}n^3$	**A1**	Simplifies by collecting terms.
		3	
2(b)	$\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}=\frac{(r+1)\left(r^2+1\right)-r\left(r^2+2r+2\right)}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}$	**M1 A1**	Puts over a common denominator and expands, AG.
	$\sum_{r=1}^{n}\frac{1-r-r^2}{\left(r^2+2r+2\right)\left(r^2+1\right)}=\sum_{r=1}^{n}\left(\frac{r+1}{(r+1)^2+1}-\frac{r}{r^2+1}\right)$ $=\frac{2}{5}-\frac{1}{2}+\frac{3}{10}-\frac{2}{5}+\frac{4}{17}-\frac{3}{10}+\cdots+\frac{n+1}{(n+1)^2+1}-\frac{n}{n^2+1}$	**M1 A1**	Shows at least three complete terms including first and last. Cancellation may be implicit.
	$=-\frac{1}{2}+\frac{n+1}{(n+1)^2+1}$	**A1**	ISW
		5	
2(c)	$-\frac{1}{2}$	**B1 FT**	FT from their answer to part **(b)**.
		1	

Q43.

1(a)	$\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin(r+1)\cos r-\cos(r+1)\sin r}{\cos(r+1)\cos r}=\frac{\sin(r+1-r)}{\cos(r+1)\cos r}$ OR $\frac{\sin(r+1)}{\cos(r+1)}-\frac{\sin r}{\cos r}=\frac{\sin r\cos 1+\cos r\sin 1}{\cos(r+1)}-\frac{\sin r}{\cos r}$ $=\frac{\sin r\cos r\cos 1+\cos^2 r\sin 1-\sin r\cos r\cos 1+\sin^2 r\sin 1}{\cos(r+1)\cos r}$	**M1**	Applies all relevant correct addition formulae from MF19 for the route chosen and combines into a single fraction.
	$=\frac{\sin 1}{\cos(r+1)\cos r}$ $\frac{\tan r+\tan 1}{1-\tan r\tan 1}-\tan r=\frac{\tan r+\tan 1-\tan r+\tan^2 r\tan 1}{1-\tan r\tan 1}$ $=\frac{\tan 1\sec^2 r}{1-\tan r\tan 1}=\frac{\tan 1}{\cos r(\cos r-\sin r\tan 1)}=\frac{\sin 1}{\cos r(\cos r\cos 1-\sin r\sin 1)}$ $=\frac{\sin 1}{\cos r\cos(r+1)}$	**A1**	SC B2 for this route completely correct to AG.
		2	
1(b)	$[\sin 1]\sum_{r=1}^{n}u_r=\tan 2-\tan 1+\tan 3-\tan 2+\ldots+\tan(n+1)-\tan n$	**M1 A1**	Shows enough terms for cancellation to be clear. There must be three complete terms including first and last. Cancelling may be implied.
	$\sum_{r=1}^{n}u_r=\frac{\tan(n+1)-\tan 1}{\sin 1}$	**A1**	OE and ISW.
		3	
1(c)	$\tan(n+1)$ oscillates as $n\to\infty$ so $u_1+u_2+u_3+\ldots$ does not converge.	**B1**	States 'oscillates' or refers to diverging values of $\tan(n+1)$, or states that $\tan(n+1)$ does not tend to a limit.
		1	

3. Summation of series

Q44.

2(a)	$\sum_{r=1}^{n}\left(r^3+3r^2+2r\right)=\frac{1}{4}n^2(n+1)^2+\frac{1}{2}n(n+1)(2n+1)+n(n+1)$	**M1 A1**	Expands and substitutes formulae. At least two formulae used for M1.
	$=\frac{1}{4}n(n+1)\left(n^2+n+4n+2+4\right)=\frac{1}{4}n(n+1)(n+2)(n+3)$	**A1**	
		3	

2(b)	$\frac{1}{r(r+1)(r+2)}=\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\ \frac{1}{2}f(1)-f(2)+\frac{1}{2}f(3)$ $+\frac{1}{2}f(2)-f(3)+\frac{1}{2}f(4)$ $+\frac{1}{2}f(3)-f(4)+\frac{1}{2}f(5)$ $\vdots$ $+\frac{1}{2}f(n-1)-f(n)+\frac{1}{2}f(n+1)$ $+\frac{1}{2}f(n)-f(n+1)+\frac{1}{2}f(n+2)$	**M1 A1**	Shows enough terms for cancelation to be clear. Where $f(x)=\frac{1}{r}$
	$=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}$	**A1**	ISW
		5	
2(c)	$\frac{1}{4}$	**B1 FT**	FT from *their* answer to part **(b)**.
		1	

Q45.

3(a)	$\sum_{r=1}^{n}\left(\ln r-2\ln(r+1)+\ln(r+2)\right)$	**B1**	Separates logarithms into correct form using a difference. Or as logarithm of product.
	$\ln 1-2\ln 2+\ln 3$ $\ln 2-2\ln 3+\ln 4$ $\ln 3-2\ln 4+\ln 5$ $\vdots$ $\ln(n-1)-2\ln n+\ln(n+1)$ $\ln n-2\ln(n+1)+\ln(n+2)$	**M1 A1**	Shows enough terms to make cancellation clear.
	$[\ln 1]-\ln 2-\ln(n+1)+\ln(n+2)=\ln\frac{n+2}{2(n+1)}.$	**A1**	AG
		4	

3(b)	$S=-\ln 2$	**B1**	States sum to infinity. AEF
	$S_n-S=\ln\left(\frac{n+2}{n+1}\right)<0.01$ leading to $n+2<e^{0.01}(n+1)$	**M1**	Forms inequality
	Least value of n is 99	**A1**	CAO
		3	

3. Summation of series

Q46.

1(a)	$\frac{1}{(ar+1)(ar+a+1)}=\frac{1}{a}\left(\frac{1}{ar+1}-\frac{1}{ar+a+1}\right)$	**M1 A1**	Finds partial fractions.
	$\sum_{r=1}^{n}\frac{1}{(ar+1)(ar+a+1)}=\frac{1}{a}\left(\frac{1}{a+1}-\frac{1}{2a+1}+\frac{1}{2a+1}-\frac{1}{3a+1}+\ldots+\frac{1}{an+1}-\frac{1}{an+a+1}\right)$	**M1**	Writes at least three complete terms, including first and last.
	$\sum_{r=1}^{n}\frac{1}{(ar+1)(ar+a+1)}=\frac{1}{a}\left(\frac{1}{a+1}-\frac{1}{an+a+1}\right)$	**A1**	OE ISW
		4	
1(b)	$\sum_{r=1}^{\infty}\frac{1}{(ar+1)(ar+a+1)}=\frac{1}{a^2+a}=\frac{1}{6}$ leading to $a^2+a-6=0$	**M1 A1**	Finds sum to infinity, forms quadratic.
	$a=2$	**A1**	CAO
		3	

Q47.

4(a)	$\sum_{r=1}^{n}e^{rx}\left(e^{2x}-2e^{x}+1\right)=e^{3x}-2e^{2x}+e^{x}$ $+e^{4x}-2e^{3x}+e^{2x}$ $+e^{5x}-2e^{4x}+e^{3x}$ $\vdots$ $+e^{(n+1)x}-2e^{nx}+e^{(n-1)x}$ $+e^{(n+2)x}-2e^{(n+1)x}+e^{nx}$	**M1 A1**	Shows enough complete terms to make pattern of cancelling clear GP method. $\left(e^{2x}-2e^{x}+1\right)\sum_{r=1}^{n}e^{rx}=\left(e^{2x}-2e^{x}+1\right)e^{x}\frac{(e^{x})^{n}-1}{e^{x}-1}$
	$=e^{x}-e^{2x}-e^{(n+1)x}+e^{(n+2)x}$	**A1**	OE $e^{x}(e^{x}-1)(e^{nx}-1)$
		3	
4(b)	$x<0$	**B1**	Accept $x\leqslant 0$.
	$e^{nx}\to 0$ as $n\to\infty$ leading to $u_1+u_2+u_3+\ldots=e^{x}-e^{2x}$	**M1 A1**	Must see $e^{nx}\to 0$ [as $n\to\infty$] agreeing with their set of x
		3	
4(c)	$\sum_{r=1}^{n}\ln u_r=\sum_{r=1}^{n}(rx+\ln(e^{x}-1)^2)$	**M1***	Uses laws of logarithms correctly.
	$\sum_{r=1}^{n}\ln u_r=\frac{1}{2}xn(n+1)+n\ln(e^{x}-1)^2$	**dM1 A1**	Applies $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$. AEF for $+n\ln(e^{x}-1)^2$
		3	

3. Summation of series

Q48.

3(a)	$8r^3+12r^2+6r+1-\left(8r^3-12r^2+6r-1\right)=24r^2+2$	**M1**	Expands.
	$24\sum_{r=1}^{n}r^2+2n=(2n+1)^3-1$	**M1 A1 A1**	Sums both sides and uses method of differences with sufficient complete terms.
	$24\sum_{r=1}^{n}r^2=8n^3+12n^2+4n=4n(n+1)(2n+1)$	**A1**	AG.
		5	

3(b)	$S_{2n}=\sum_{r=1}^{2n}r^2+2\sum_{r=1}^{n}(2r)^2=\sum_{r=1}^{2n}r^2+8\sum_{r=1}^{n}r^2$	**M1**	Relates with sum of squares.
	$S_{2n}=\frac{1}{6}(2n)(2n+1)(4n+1)+\frac{8}{6}n(n+1)(2n+1)$	**M1**	Applies $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$.
	$S_{2n}=\frac{1}{3}n(2n+1)(4n+1+4n+4)=\frac{1}{3}n(2n+1)(8n+5)$	**A1**	$\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$.
	Alternative for 3(b)		
	$S_{2n}=\sum_{r=1}^{n}(2r-1)^2+3\sum_{r=1}^{n}(2r)^2=\sum_{r=1}^{n}\left(4r^2-4r+1+12r^2\right)$	**M1**	Relates with sum of squares.
	$\frac{16n(n+1)(2n+1)}{6}-4\frac{n(n+1)}{2}+n$	**M1**	Applies $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$
	$\frac{n}{3}\left(16n^2+18n+5\right)=\frac{n}{3}(2n+1)(8n+5)$	**A1**	
		3	
3(c)	$\frac{16}{3}$	**B1 FT**	
		1	

3. Summation of series

Q49.

1(a)	$\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2r = \frac{1}{6}n(n+1)(2n+1) + n(n+1)$	**M1**	Uses List of formulae (MF19).
	$\frac{1}{6}n(n+1)(2n+7)$	**A1**	At least all like terms collected and fractions simplified.
		2	
1(b)	$\frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$	**M1 A1**	Expresses in partial fractions.
	$\sum_{r=1}^{n} \frac{2}{r(r+2)} = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} \vdots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}$	**M1**	Writes enough terms.
	$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$	**A1**	OE
		4	
1(c)	$\frac{3}{4}$	**B1 FT**	FT *their* part **(b)** of correct form.
		1	

4. Matrices

Q1.

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & k \\ -3 & -4 \end{pmatrix}.$$

(a) Find the value of k for which $\mathbf{A}$ is singular. [2]

It is now given that $k = 6$ so that $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$.

(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by $\mathbf{A}$. [6]

(c) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

(i) Given that the area of triangle DEF is $10\,\text{cm}^2$, find the area of triangle PQR. [2]

(ii) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

9231/01/SP/20 Q5

Q2.

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

9231/11/M/J/20 Q6

4. Matrices

Q3.

4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

9231/13/M/J/20 Q4

Q4.

1 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x+3y=0$ is invariant under the transformation represented by **M**.

(c) Find the values of a and b. [5]

9231/11/O/N/20 Q1

4. Matrices

Q5.

4 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [1]

(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [3]

(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

9231/12/O/N/20 Q4

Q6.

4 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. [5]

9231/11/M/J/21 Q4

4. Matrices

Q7.

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

(b) Given that **A** is singular, find the value of k. [3]

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

9231/13/M/J/21 Q4

Q8.

The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by **M** has exactly one invariant line through the origin, giving your answers in terms of π. [9]

9231/11/O/N/21 Q4

4. Matrices

Q9.

(a) Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. [2]

(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. [4]

9231/12/O/N/21 Q1

Q10.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which $\mathbf{A}$ is non-singular. [3]

(b) Given that $\mathbf{A}$ is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

9231/11/M/J/22 Q7

4. Matrices

Q11.

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

9231/13/M/J/22 Q5

Q12.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) The triangle ABC in the x–y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Find, in terms of k, the single matrix which transforms triangle DEF onto triangle ABC. [2]

(c) Find the set of values of k for which the transformation represented by $\mathbf{M}$ has no invariant lines through the origin. [7]

9231/11/O/N/22 Q5

4. Matrices

Q13.

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ or 1.

(a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

(c) Show that the invariant points of the transformation represented by $\mathbf{M}$ lie on the line $y = \frac{k^2}{1-k}x$. [4]

(d) The triangle ABC in the x-y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Find the value of k for which the area of triangle DEF is equal to the area of triangle ABC. [2]

9231/12/O/N/22 Q3

Solutions

Q1.

5(a)	$-20 + 3k = 0$	1	**M1**
	$k = \frac{20}{3}$	1	**A1**
		2	
5(b)	Consider $\begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+6y \\ -3x-4y \end{pmatrix}$ $\left(= \begin{pmatrix} X \\ Y \end{pmatrix}\right)$	1	**M1**
	Line through O, so $y = mx$	1	**M1**
	Invariant line, so $Y = mX$, $\Rightarrow -3x - 4mx = m(5x + 6mx)$	2	**M1A1**
	$6m^2 + 9m + 3 = 0$ $m = -1, -\frac{1}{2}$	1	**A1**
	Invariant lines $y = -x$, $2y + x = 0$	1	**A1**
		6	
5(c)(i)	det $\mathbf{A} = -20 + 18 = -2$	1	**M1**
	Area $= 2 \times 10 = 20\,\text{cm}^2$	1	**A1**
		2	
5(c)(ii)	$\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -\frac{3}{2} & -\frac{5}{2} \end{pmatrix}$	**2**	**M1A1**

4. Matrices

Q2.

6(a)	$\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1} = \frac{1}{2}$	**M1 A1**
	$d = 15$	**A1**
		3
6(b)	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix}$ so true when $n = 1$.	**B1**
	Assume that it is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$.	**B1**
	Then $\mathbf{A}^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2(2^k - 1) + 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$	**M1A1**
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	**A1**
		5
6(c)	$\mathbf{A}^n\mathbf{B} = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}$	**M1A1**
	$\begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n + 32)x \end{pmatrix}$	**B1**
	$(2^n + 32)x = 2^{n+1}x \Rightarrow 2^n = 32 \Rightarrow n = 5$	**M1 A1**
		5

Q3.

4(a)	$k\begin{vmatrix} -1 & -1 \\ 1 & -k \end{vmatrix} + 2\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 0 \Rightarrow k^2 + k + 2 = 0$	**M1 A1**
	$1 - 4(2) = -7 < 0 \Rightarrow$ Non-singular	**A1**
		3
4(b)	$\begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix}\begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -3k \\ 1 & -3 \\ -1 & 0 \end{pmatrix}$	**M1 A1**
	$= \begin{pmatrix} -2 & 9k + 3 \\ -1 & -3k - 3 \end{pmatrix} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix} \Rightarrow k = -\frac{1}{2}$	**A1**
		3
4(c)	$\begin{pmatrix} 2 & \frac{3}{2} \\ 1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + \frac{3}{2}y \\ x + \frac{3}{2}y \end{pmatrix}$	**B1**
	$x + \frac{3}{2}mx = m\left(2x + \frac{3}{2}mx\right)$	**M1 A1**
	$1 + \frac{3}{2}m = 2m + \frac{3}{2}m^2 \Rightarrow 3m^2 + m - 2 = 0$	**A1**
	$y = -x$ and $3y - 2x = 0$	**A1**
		5

4. Matrices

Q4.

1(a)	One-way stretch followed by a shear.	**B2**	Both named correctly. Award B1 if given in the wrong order.
		2	
1(b)	$\mathbf{M}^{-1}=\frac{1}{a}\begin{pmatrix}1 & -b\\ 0 & a\end{pmatrix}$	**M1 A1**	
		2	
1(c)	$a=2$	**B1**	
	$\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}ax+by\\ y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$=\begin{pmatrix}ax-\frac{1}{3}bx\\ -\frac{1}{3}x\end{pmatrix}$	**M1**	Uses $x+3y=0$.
	$x=ax-\frac{1}{3}bx\Rightarrow 1=a-\frac{1}{3}b$	**M1**	Uses that line is invariant (or $X+3Y=0$).
	$b=3$	**A1**	
		5	

Q5.

4(a)	Reflection in the line $y=x$.	**B1**	Only mention reflection (and no other transformation).
		1	
4(b)	Rotation	**B1**	Writes 'rotation' or 'rotate' (and no other transformation)
	$\frac{1}{3}\pi$ **anticlockwise** about the origin.	**B1**	Or 60°
		2	
4(c)	$\mathbf{AB}=\begin{pmatrix}\frac{1}{2}\sqrt{3} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\sqrt{3}\end{pmatrix}$ or $\det\mathbf{AB}=\det\mathbf{A}\det\mathbf{B}$	**M1**	Finds **AB** or uses product of determinants. Full marks here may be obtained by arguing that both reflection and rotation preserve [absolute] value of area and so also does their combination.
	$\det\mathbf{AB}=-1$	**B1**	
	Area of PQR = |-1| Area of DEF	**A1**	
		3	
4(d)	$(\mathbf{AB})^{-1}=-\begin{pmatrix}-\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\sqrt{3} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\sqrt{3}\end{pmatrix}$	**M1 A1**	or using $(\mathbf{AB})^{-1}=\mathrm{B}^{-1}\mathrm{A}^{-1}$
		2	

4. Matrices

Q6.

4(a)	$\begin{pmatrix}\cos 60 & -\sin 60\\ \sin 60 & \cos 60\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	**B1**	Rotation 60° anticlockwise about the origin.
	$\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}$	**B1**	One-way stretch in the x-direction, scale factor d.
	$\mathbf{M}=\begin{pmatrix}d & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{d}{2} & -\frac{d\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	**M1 A1**	Correct order.
		4	
4(b)	$d=\frac{1}{2}d^2$	**M1**	Uses value of $\det\mathbf{M}$.
	$d\neq 0\Rightarrow d=2$	**A1**	AG
		2	
4(c)	$\mathbf{M}^{-1}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}$	**B1 FT**	Inverse of *their* **M**.
	$\mathbf{N}=\mathbf{M}^{-1}\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\frac{1}{2} & \sqrt{3}\\ -\frac{\sqrt{3}}{2} & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$	**M1**	Multiplies on the left by the inverse of *their* **M**.
	$=\frac{1}{4}\begin{pmatrix}1+\sqrt{3} & 1+\sqrt{3}\\ 1-\sqrt{3} & 1-\sqrt{3}\end{pmatrix}$	**A1**	CAO
		3	
4(d)	$\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+y\\ \frac{1}{2}x+\frac{1}{2}y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$\frac{1}{2}x+\frac{1}{2}mx=m\left(x+mx\right)$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$1+m=2m+2m^2\Rightarrow 2m^2+m-1=0$	**A1**	
	$y=\frac{1}{2}x$ and $y=-x$	**A1**	WWW
		5	

4. Matrices

Q7.

4(a)	$\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2&k&k\\5&-1&3\\1&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2+k&k\\8&-1\\2&0\end{pmatrix}$ Or $\begin{pmatrix}0&1&1\\-1&2&0\end{pmatrix}\begin{pmatrix}2&k&k\\5&-1&3\\1&0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}6&-1&4\\8&-k-2&-k+6\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}$	**M1 A1**	Multiplies two matrices correctly.
	$\begin{pmatrix}10&-1\\-k+14&-k-2\end{pmatrix}$	**A1**	
		3	
4(b)	$2\begin{vmatrix}-1&3\\0&1\end{vmatrix}-k\begin{vmatrix}5&3\\1&1\end{vmatrix}+k\begin{vmatrix}5&-1\\1&0\end{vmatrix}=0$ leading to $-2-2k+k=0$	**M1 A1**	Sets determinant equal to zero and forms linear equation.
	$k=-2$	**A1**	
		3	

4(c)	$\begin{pmatrix}10&-1\\16&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10x-y\\16x\end{pmatrix}$	**M1**	Transforms $\begin{pmatrix}x\\y\end{pmatrix}$. Allow $q\begin{pmatrix}1\\m\end{pmatrix}$ where q is x, t or a nonzero number.
	$10x-mx=X$ and $16x=mX$	**M1 A1**	Uses $y=mx$ and $Y=mX$. Expect $16x=m(10x-mx)$.
	$16=10m-m^2$ $\quad[m^2-10m+16=0]$	**A1**	OE
	$y=2x$ and $y=8x$	**A1**	
		5	

4. Matrices

Q8.

4(a)	One-way stretch [scale factor 3 parallel to the x-axis] followed by a rotation, [anticlockwise centred at the origin, through an angle θ.]	**B2**	Award B1 if given in the wrong order.
		2	
4(b)	$\mathbf{M}=\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}$	**B1**	
	$\begin{pmatrix}3\cos\theta & -\sin\theta\\ 3\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x\cos\theta-y\sin\theta\\ 3x\sin\theta+y\cos\theta\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$3x\sin\theta+mx\cos\theta=3mx\cos\theta-m^2x\sin\theta$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$m^2\sin\theta-2m\cos\theta+3\sin\theta=0$	**A1**	
	$4\cos^2\theta-12\sin^2\theta=0$	**M1 A1**	Sets discriminant equal to 0.
	$\tan\theta=\pm\frac{1}{\sqrt{3}}$ or $4\cos^2\theta-12\left(1-\cos^2\theta\right)=0$ leading to $16\cos^2\theta-12=0$ leading to $\cos\theta=\pm\frac{\sqrt{3}}{2}$	**M1**	Uses an appropriate trigonometric identity.
	$\theta=\frac{1}{6}\pi$ and $\theta=\frac{5}{6}\pi$	**A1**	Allow 30° and 150°
		9	

Q9.

1(a)	Enlargement, scale factor 6.	**B1**	
		1	
1(b)	$\det\mathbf{A}=6-8=-2$	**M1**	Finds $\det\mathbf{A}$.
	Area $=2\times13=26\,\text{cm}^2$	**A1**	
		2	
1(c)	$\mathbf{A}^{-1}=-\frac{1}{2}\begin{pmatrix}2 & -4\\ -2 & 3\end{pmatrix}$	**M1**	Finds $\mathbf{A}^{-1}$.
	$\mathbf{B}=-3\begin{pmatrix}2 & -4\\ -2 & 3\end{pmatrix}=\begin{pmatrix}-6 & 12\\ 6 & -9\end{pmatrix}$	**A1**	AEF Could be solved by equations.
		2	
1(d)	$\begin{pmatrix}3 & 4\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x+4y\\ 2x+2y\end{pmatrix}$	**B1**	Finds $\begin{pmatrix}X\\ Y\end{pmatrix}$
	$\left.\begin{matrix}3x+4y=x\\ 2x+2y=y\end{matrix}\right\}$ leading to $\begin{cases}2x+4y=0\\ 2x+y=0\end{cases}$	**M1**	Uses $\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}X\\ Y\end{pmatrix}$ to form simultaneous equations.
	$y=-2x$ leading to $-6x=0$ leading to $x=0, y=0$	**M1 A1**	Solves equations or states that $\det\begin{pmatrix}2 & 4\\ 2 & 1\end{pmatrix}\neq0$, AG.
		4	

4. Matrices

Q10.

7(a)	$\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & k \\ 7 & 8 \end{vmatrix} = 0 \Rightarrow -12k + 60 = 0$	**M1 A1**	Sets $\det \mathbf{A} = 0$. SOI
	$k \neq 5$	**A1**	
		3	
7(b)	$\frac{1}{60-12k}\begin{vmatrix} k & 6 \\ 8 & 9 \end{vmatrix} = \frac{9k-48}{60-12k} = \frac{3k-16}{4(5-k)}$	**M1 A1**	Finds one correct entry.
	$\frac{-1}{60-12k}\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = \frac{6}{60-12k} = \frac{1}{2(5-k)}$	**A1**	
	$\frac{1}{60-12k}\begin{vmatrix} 2 & 3 \\ k & 6 \end{vmatrix} = \frac{12-3k}{60-12k} = \frac{4-k}{4(5-k)}$	**A1**	
		4	

7(c)	$\mathbf{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix}$	**M1 A1**	Finds **BA**.
	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a+2b+3c & d+2e+3f \\ 4a+kb+6c & 4d+ke+6f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$	**M1**	Sets **C** as 3×2 matrix.
	$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	**A1**	Accept any a, b, c, d, e and f such that $\begin{pmatrix} a+2b+3c & d+2e+3f \\ 4a+kb+6c & 4d+ke+6f \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$.
		4	
7(d)	$\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ kx+4y \end{pmatrix}$	**B1**	Transforms $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$.
	$kx + 4mx = m(2x + mx)$	**M1 A1**	Uses $y = mx$ and $Y = mX$.
	$m^2 - 2m - k = 0$	**A1**	
	$4 + 4k > 0$	**M1**	Sets discriminant positive.
	$k > -1 \quad k \neq 8$	**A1**	
		6	

4. Matrices

Q11.

5(a)	A shear in the x-direction.	**B1**	Accept 'shear'.
		1	
5(b)	$\mathbf{A}=\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ so true when $n=1$.	**B1**	States base case.
	Assume that it is true for $n=k$, so $\mathbf{A}^k=\begin{pmatrix}1 & ka\\ 0 & 1\end{pmatrix}$.	**B1**	States inductive hypothesis.
	Then $\mathbf{A}^{k+1}=\begin{pmatrix}1 & ka\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & a+ak\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & (k+1)a\\ 0 & 1\end{pmatrix}$	**M1 A1**	Multiplies $\mathbf{A}^k$ with $\mathbf{A}$.
	If it is true for [$n=1$ and] $n=k$ then it is also true for $n=k+1$. Hence, [by induction,] true for all positive integers.	**A1**	Everything correct and states conclusion.
		5	
5(c)	$\mathbf{A}^n\mathbf{B}=\begin{pmatrix}1 & na\\ 0 & 1\end{pmatrix}\begin{pmatrix}b & b\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}=\begin{pmatrix}b+n & b+n\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}$	**M1 A1**	Uses formula given in part **(b)**.
	$\begin{pmatrix}b+n & b+n\\ \frac{1}{a} & \frac{1}{a}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}(b+n)(x+y)\\ \frac{1}{a}(x+y)\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ by multiplying matrices to find $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$\frac{1}{a}(1+m)=m(b+n)(1+m)$	**M1**	Uses $y=mx$ and $Y=mX$
	$y=-x$	**A1**	
	$y=\frac{1}{a(b+n)}x$	**A1**	
		6	

4. Matrices

Q12.

5(a)	Shear (in the x-direction) followed by a rotation (anticlockwise about the origin through 45°).	**B2**	Award B1 if given in the wrong order.
		2	
5(b)	$\mathbf{M}^{-1}=\frac{1}{1}\begin{pmatrix}\frac{k+1}{\sqrt{2}} & -\frac{k-1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{k+1}{\sqrt{2}} & -\frac{k-1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$	**M1 A1**	Finds $\mathbf{M}^{-1}$.
		2	
5(c)	$\mathbf{M}=\begin{pmatrix}\frac{1}{2}\sqrt{2} & \frac{1}{2}(k-1)\sqrt{2}\\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2}\end{pmatrix}$	**B1**	
	$\frac{1}{2}\sqrt{2}\begin{pmatrix}1 & k-1\\ 1 & k+1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{2}\sqrt{2}\begin{pmatrix}x+(k-1)y\\ x+(k+1)y\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$.
	$x+m(k+1)x=m(x+m(k-1)x)$	**M1 A1**	Uses $y=mx$ and $Y=mX$.
	$(k-1)m^2-km-1=0$	**A1**	
	$k^2+4k-4<0$	**M1**	Sets discriminant negative.
	$2(-1-\sqrt{2})<k<2(-1+\sqrt{2})$	**A1**	

5(c)	**Alternative method for question 5(c)**		
	$\mathbf{M}=\begin{pmatrix}\frac{1}{2}\sqrt{2} & \frac{1}{2}(k-1)\sqrt{2}\\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2}\end{pmatrix}$	**B1**	
	$[\mathbf{M}-\lambda\mathbf{I}=]\begin{vmatrix}\frac{1}{2}\sqrt{2}-\lambda & \frac{1}{2}(k-1)\sqrt{2}\\ \frac{1}{2}\sqrt{2} & \frac{1}{2}(k+1)\sqrt{2}-\lambda\end{vmatrix}=0$	**B1**	Sets up to find characteristic equation.
	$(\frac{1}{2}\sqrt{2}-\lambda)(\frac{1}{2}(k+1)\sqrt{2}-\lambda)-\frac{1}{2}(k-1)\sqrt{2}\frac{1}{2}\sqrt{2}=0$	**M1 A1**	Evaluates determinant.
	$2\lambda^2-\sqrt{2}(k+2)\lambda+2=0$	**A1**	
	$k^2+4k-4<0$	**M1**	Sets discriminant negative.
	$2(-1-\sqrt{2})<k<2(-1+\sqrt{2})$	**A1**	
		7	

4. Matrices

Q13.

3(a)	Shear [in the y-direction] followed by a stretch, [parallel to the y-axis, scale factor k].	**B2**	Award B1 if given in the wrong order.
		2	
3(b)	$\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & 0\\ -k & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}^{-1}=\begin{pmatrix}1 & 0\\ 0 & \frac{1}{k}\end{pmatrix}$	**B1**	
	$\mathbf{M}^{-1}=\begin{pmatrix}1 & 0\\ -k & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{k}\end{pmatrix}$	**B1**	Correct order.
		2	
3(c)	$\mathbf{M}=\begin{pmatrix}1 & 0\\ k^2 & k\end{pmatrix}$	**B1**	
	$\begin{pmatrix}1 & 0\\ k^2 & k\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ k^2x+ky\end{pmatrix}$	**B1**	Transforms $\begin{pmatrix}x\\ y\end{pmatrix}$ to $\begin{pmatrix}X\\ Y\end{pmatrix}$
	$k^2x+ky=y$	**M1**	Sets $\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$
	$y=\frac{k^2}{1-k}x$	**A1**	AG.
		4	
3(d)	$1=\lvert k\rvert$	**M1**	Uses $\lvert DEF\rvert=\lvert\det\mathbf{M}\rvert\lvert ABC\rvert$
	$k=-1$	**A1**	
		2	

5. Polar coordinates

Q1.

4 The curves C_1 and C_2 have polar equations

$$r = \theta + 2 \quad \text{and} \quad r = \theta^2$$

respectively, where $0 \leqslant \theta \leqslant \pi$.

(i) Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

(ii) Sketch C_1 and C_2 on the same diagram. [2]

(iii) Find the area bounded by C_1, C_2 and the line $\theta = 0$. [3]

9231/01/M/J/08 Q4

Q2.

3 The curve C has polar equation

$$r = \left(\tfrac{1}{2}\pi - \theta\right)^2,$$

where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [3]

Find the area of the region bounded by C and the initial line, leaving your answer in terms of π. [3]

9231/01/O/N/08 Q3

Q3.

5 Draw a sketch of the curve C whose polar equation is $r = \theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. [2]

On the same diagram draw the line $\theta = \alpha$, where $0 < \alpha < \frac{1}{2}\pi$. [1]

The region bounded by C and the line $\theta = \frac{1}{2}\pi$ is denoted by R. Find the exact value of α for which the line $\theta = \alpha$ divides R into two regions of equal area. [4]

9231/01/M/J/09 Q5

5. Polar coordinates

Q4.

10 The curve C has polar equation

$$r = a\sin 3\theta,$$

where $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(i) Show that the area of the region enclosed by C is $\frac{1}{12}\pi a^2$. [3]

(ii) Show that, at the point of C at maximum distance from the initial line,

$$\tan 3\theta + 3\tan\theta = 0.$$ [3]

(iii) Use the formula

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

to find this maximum distance. [4]

(iv) Draw a sketch of C. [2]

9231/01/O/N/09 Q10

Q5.

2 The curve C has polar equation

$$r = a\left(1 - \mathrm{e}^{-\theta}\right),$$

where a is a positive constant and $0 \leqslant \theta < 2\pi$.

(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by C and the lines $\theta = \ln 2$ and $\theta = \ln 4$ is

$$\tfrac{1}{2}a^2\left(\ln 2 - \tfrac{13}{32}\right).$$ [4]

9231/11/M/J/10 Q2

5. Polar coordinates

Q6.

11 The curve C has polar equation

$$r = \frac{a}{1+\theta},$$

where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Show that r decreases as θ increases. [2]

(ii) The point P of C is further from the initial line than any other point of C. Show that, at P,

$$\tan\theta = 1 + \theta,$$

and verify that this equation has a root between 1.1 and 1.2. [4]

(iii) Draw a sketch of C. [3]

(iv) Find the area of the region bounded by the initial line, the line $\theta = \frac{1}{2}\pi$ and C, leaving your answer in terms of π and a. [3]

9231/13/M/J/10 Q11

Q7.

8 The curves C_1 and C_2 have polar equations given by

$$C_1: \quad r = 3\sin\theta, \qquad 0 \leqslant \theta < \pi,$$
$$C_2: \quad r = 1 + \sin\theta, \quad -\pi < \theta \leqslant \pi.$$

(i) Find the polar coordinates of the points, other than the pole, where C_1 and C_2 meet. [2]

(ii) In a single diagram, draw sketch graphs of C_1 and C_2. [3]

(iii) Show that the area of the region which is inside C_1 but outside C_2 is π. [5]

9231/01/O/N/10 Q8

Q8.

5 The curve C has polar equation $r = 2\cos 2\theta$. Sketch the curve for $0 \leqslant \theta < 2\pi$. [4]

Find the exact area of one loop of the curve. [4]

9231/11/M/J/11 Q5

5. Polar coordinates

Q9.

6 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = a,$$
$$C_2: \quad r = 2a\cos 2\theta, \text{ for } 0 \leqslant \theta \leqslant \tfrac{1}{4}\pi,$$

where a is a positive constant. Sketch C_1 and C_2 on the same diagram. [3]

The curves C_1 and C_2 intersect at the point with polar coordinates (a, β). State the value of β. [1]

Show that the area of the region bounded by the initial line, the arc of C_1 from $\theta = 0$ to $\theta = \beta$, and the arc of C_2 from $\theta = \beta$ to $\theta = \frac{1}{4}\pi$ is

$$a^2\left(\tfrac{1}{6}\pi - \tfrac{1}{8}\sqrt{3}\right).$$ [4]

9231/13/M/J/11 Q6

Q10.

10 The curve C has polar equation $r = 3 + 2\cos\theta$, for $-\pi < \theta \leqslant \pi$. The straight line l has polar equation $r\cos\theta = 2$. Sketch both C and l on a single diagram. [3]

Find the polar coordinates of the points of intersection of C and l. [4]

The region R is enclosed by C and l, and contains the pole. Find the area of R. [6]

9231/11/O/N/11 Q10

Q11.

8 The curve C has polar equation $r = 1 + \sin\theta$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

The area of the region enclosed by the initial line, the half-line $\theta = \frac{1}{2}\pi$, and the part of C for which θ is positive, is denoted by A_1. The area of the region enclosed by the initial line, and the part of C for which θ is negative, is denoted by A_2. Find the ratio $A_1 : A_2$, giving your answer correct to 1 decimal place. [8]

9231/13/O/N/11 Q8

Q12.

4 The curve C has polar equation $r = 2 + 2\cos\theta$, for $0 \leqslant \theta \leqslant \pi$. Sketch the graph of C. [2]

Find the area of the region R enclosed by C and the initial line. [4]

The half-line $\theta = \frac{1}{5}\pi$ divides R into two parts. Find the area of each part, correct to 3 decimal places. [3]

9231/11/M/J/12 Q4

5. Polar coordinates

Q13.

The curve C has cartesian equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

where a is a positive constant. Show that C has polar equation

$$r^2 = a^2 \cos 2\theta. \quad [2]$$

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area of the sector between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. [3]

Find the polar coordinates of all points of C where the tangent is parallel to the initial line. [7]

9231/13/M/J/12 Q11

Q14.

1 Find the cartesian equation corresponding to the polar equation $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$. [3]

Sketch the the graph of $r = (\sqrt{2})\sec(\theta - \frac{1}{4}\pi)$, for $-\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line. [2]

9231/11/O/N/12 Q1

Q15.

5 The curve C has polar equation $r = 1 + 2\cos\theta$. Sketch the curve for $-\frac{2}{3}\pi \leqslant \theta < \frac{2}{3}\pi$. [2]

Find the area bounded by C and the half-lines $\theta = -\frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$. [4]

9231/13/O/N/12 Q5

Q16.

1 Find the area of the region enclosed by the curve with polar equation $r = 2(1 + \cos\theta)$, for $0 \leqslant \theta < 2\pi$. [4]

9231/11/M/J/13 Q1

Q17.

10 Use the identity $2\sin P\cos Q \equiv \sin(P + Q) + \sin(P - Q)$ to show that

$$2\sin\theta\cos(\theta - \tfrac{1}{4}\pi) \equiv \cos(2\theta - \tfrac{3}{4}\pi) + \frac{1}{\sqrt{2}}. \quad [3]$$

A curve has polar equation $r = 2\sin\theta\cos(\theta - \frac{1}{4}\pi)$, for $0 \leqslant \theta \leqslant \frac{3}{4}\pi$. Sketch the curve and state the polar equation of its line of symmetry, justifying your answer. [3]

Show that the area of the region enclosed by the curve is $\frac{3}{8}(\pi + 1)$. [6]

9231/13/M/J/13 Q10

5. Polar coordinates

Q18.

1 The curve C has polar equation $r = 2e^{\theta}$, for $\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the area of the region bounded by the half-lines $\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{2}\pi$ and C, [2]

(ii) the length of C. [3]

9231/11/O/N/13 Q1

Q19.

10 The curve C has polar equation $r = 2\sin\theta(1 - \cos\theta)$, for $0 \leqslant \theta \leqslant \pi$. Find $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ and hence find the polar coordinates of the point of C that is furthest from the pole. [5]

Sketch C. [2]

Find the exact area of the sector from $\theta = 0$ to $\theta = \frac{1}{4}\pi$. [6]

9231/13/O/N/13 Q10

Q20.

5 The curve C has polar equation $r = a(1 + \sin\theta)$, where a is a positive constant and $0 \leqslant \theta < 2\pi$. Draw a sketch of C. [2]

Find the exact value of the area of the region enclosed by C and the half-lines $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$. [4]

9231/11/M/J/14 Q5

Q21.

4 The curve C has cartesian equation $(x^2 + y^2)^2 = 2a^2xy$, where a is a positive constant. Show that the polar equation of C is $r^2 = a^2\sin 2\theta$. [3]

Sketch C for $-\pi < \theta \leqslant \pi$. [2]

Find the area enclosed by one loop of C. [2]

9231/13/M/J/14 Q4

Q22.

8 A circle has polar equation $r = a$, for $0 \leqslant \theta < 2\pi$, and a cardioid has polar equation $r = a(1 - \cos\theta)$, for $0 \leqslant \theta < 2\pi$, where a is a positive constant. Draw sketches of the circle and the cardioid on the same diagram. [3]

Write down the polar coordinates of the points of intersection of the circle and the cardioid. [2]

Show that the area of the region that is both inside the circle and inside the cardioid is

$$(\tfrac{5}{4}\pi - 2)a^2.$$ [6]

9231/11/O/N/14 Q8

5. Polar coordinates

Q23.

5 The curves C_1 and C_2 have polar equations

$$C_1: \quad r = \frac{1}{\sqrt{2}}, \quad \text{for } 0 \leqslant \theta < 2\pi,$$
$$C_2: \quad r = \sqrt{(\sin \tfrac{1}{2}\theta)}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

Sketch C_1 and C_2 on the same diagram. [3]

Find the exact value of the area of the region enclosed by C_1, C_2 and the half-line $\theta = 0$. [4]

9231/11/M/J/15 Q5

Q24.

2 The curve C has polar equation $r = e^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [6]

9231/13/M/J/15 Q2

Q25.

The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2 \sin^2(\tfrac{1}{2}\theta),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]

9231/11/O/N/15 Q11

Q26.

4 A curve C has polar equation $r^2 = 8\,\text{cosec}\,2\theta$ for $0 < \theta < \frac{1}{2}\pi$. Find a cartesian equation of C. [3]

Sketch C. [2]

Determine the exact area of the sector bounded by the arc of C between $\theta = \frac{1}{6}\pi$ and $\theta = \frac{1}{3}\pi$, the half-line $\theta = \frac{1}{6}\pi$ and the half-line $\theta = \frac{1}{3}\pi$. [3]

[It is given that $\int \text{cosec}\, x\, dx = \ln\left|\tan \tfrac{1}{2}x\right| + c$.]

9231/11/M/J/16 Q4

5. Polar coordinates

Q27.

7 A curve has polar equation $r = \dfrac{1}{1-\cos\theta}$, for $0 < \theta < 2\pi$. Find, in the form $y^2 = \mathrm{f}(x)$, the cartesian equation of the curve. [3]

Hence sketch the curve, and shade the region whose area is given by $\frac{1}{2}\displaystyle\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{1}{(1-\cos\theta)^2}\,\mathrm{d}\theta$. [3]

Using the cartesian equation of the curve, find the area of this region. [3]

9231/13/M/J/16 Q7

Q28.

A curve C has parametric equations

$$x = 1 - 3t^2, \quad y = t(1-3t^2), \quad \text{for } 0 \leqslant t \leqslant \frac{1}{\sqrt{3}}.$$

Show that $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1+9t^2)^2$. [2]

Hence find

(i) the arc length of C, [2]

(ii) the surface area generated when C is rotated through 2π radians about the x-axis. [3]

Use the fact that $t = \dfrac{y}{x}$ to find a cartesian equation of C. Hence show that the polar equation of C is $r = \sec\theta(1 - 3\tan^2\theta)$, and state the domain of θ. [4]

Find the area of the region enclosed between C and the initial line. [3]

9231/11/O/N/16 Q11

5. Polar coordinates

Q29.

11 The curve C has polar equation $r = a(1 + \sin\theta)$ for $-\pi < \theta \leqslant \pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C. [4]

(iii) Show that the length of the arc of C from the pole to the point furthest from the pole is given by

$$s = (\sqrt{2})a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sqrt{(1 + \sin\theta)}\, \mathrm{d}\theta.$$ [3]

(iv) Show that the substitution $u = 1 + \sin\theta$ reduces this integral for s to $(\sqrt{2})a \int_0^2 \frac{1}{\sqrt{(2-u)}}\, \mathrm{d}u$. Hence evaluate s. [4]

9231/11/M/J/17 Q11

Q30.

A curve C has polar equation $r = 2a\cos\left(2\theta + \frac{1}{2}\pi\right)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Show that $r = -2a\sin 2\theta$ and sketch C. [4]

(ii) Deduce that the cartesian equation of C is

$$(x^2 + y^2)^{\frac{3}{2}} = -4axy.$$ [2]

(iii) Find the area of one loop of C. [5]

(iv) Show that, at the points (other than the pole) at which a tangent to C is parallel to the initial line,

$$2\tan\theta = -\tan 2\theta.$$ [3]

9231/13/M/J/17 Q11

Q31.

The polar equation of a curve C is $r = a(1 + \cos\theta)$ for $0 \leqslant \theta < 2\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Show that the cartesian equation of C is

$$x^2 + y^2 = a\left(x + \sqrt{(x^2 + y^2)}\right).$$ [2]

(iii) Find the area of the sector of C between $\theta = 0$ and $\theta = \frac{1}{3}\pi$. [4]

(iv) Find the arc length of C between the point where $\theta = 0$ and the point where $\theta = \frac{1}{3}\pi$. [5]

9231/11/O/N/17 Q11

5. Polar coordinates

Q32.

3 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Find a cartesian equation of C. [3]

9231/11/M/J/18 Q3

Q33.

8 The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \pi$, as follows:

$$C_1: r = a,$$
$$C_2: r = 2a|\cos\theta|,$$

where a is a positive constant. The curves intersect at the points P_1 and P_2.

(i) Find the polar coordinates of P_1 and P_2. [2]

(ii) In a single diagram, sketch C_1, C_2 and their line of symmetry. [3]

(iii) The region R enclosed by C_1 and C_2 is bounded by the arcs OP_1, P_1P_2 and P_2O, where O is the pole. Find the area of R, giving your answer in exact form. [5]

9231/13/M/J/18 Q8

Q34.

9 The curve C has polar equation

$$r = 5\sqrt{(\cot\theta)},$$

where $0.01 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Find the area of the finite region bounded by C and the line $\theta = 0.01$, showing full working. Give your answer correct to 1 decimal place. [3]

Let P be the point on C where $\theta = 0.01$.

(ii) Find the distance of P from the initial line, giving your answer correct to 1 decimal place. [2]

(iii) Find the maximum distance of C from the initial line. [3]

(iv) Sketch C. [2]

9231/11/O/N/18 Q9

5. Polar coordinates

Q35.

3 The curve C has polar equation $r = a\cos 3\theta$, for $-\frac{1}{6}\pi \leqslant \theta \leqslant \frac{1}{6}\pi$, where a is a positive constant.

(i) Sketch C. [2]

(ii) Find the area of the region enclosed by C, showing full working. [3]

(iii) Using the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find a cartesian equation of C. [3]

9231/12/O/N/18 Q3

Q36.

11 Answer only **one** of the following two alternatives.

EITHER

The curve C_1 has polar equation $r^2 = 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

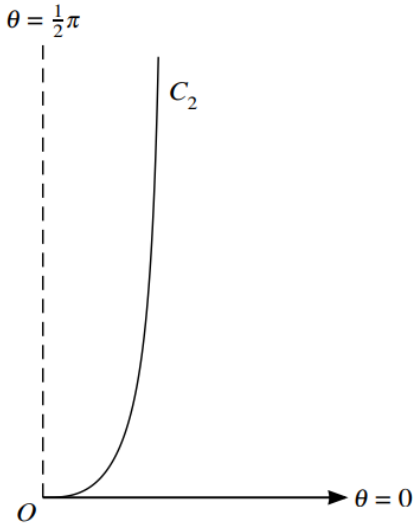
$$2\theta\tan\theta = 1$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r^2 = \theta\sec^2\theta$, for $0 \leqslant \theta < \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(ii) Find the exact value of θ at Q. [2]

(iii) The diagram below shows the curve C_2. Sketch C_1 on this diagram. [2]



(iv) Find, in exact form, the area of the region OPQ enclosed by C_1 and C_2. [5]

9231/11/M/J/19 Q11

5. Polar coordinates

Q37.

2 The curve C has polar equation $r^2 = \ln(1 + \theta)$, for $0 \leqslant \theta \leqslant 2\pi$.

(i) Sketch C. [2]

(ii) Using the substitution $u = 1 + \theta$, or otherwise, find the area of the region bounded by C and the initial line, leaving your answer in an exact form. [5]

9231/13/M/J/19 Q2

Q38.

The curves C_1 and C_2 have polar equations, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, as follows:

$$C_1 : r = 2(e^{\theta} + e^{-\theta}),$$
$$C_2 : r = e^{2\theta} - e^{-2\theta}.$$

The curves intersect at the point P where $\theta = \alpha$.

(i) Show that $e^{2\alpha} - 2e^{\alpha} - 1 = 0$. Hence find the exact value of α and show that the value of r at P is $4\sqrt{2}$. [6]

(ii) Sketch C_1 and C_2 on the same diagram. [3]

(iii) Find the area of the region enclosed by C_1, C_2 and the initial line, giving your answer correct to 3 significant figures. [5]

9231/11/O/N/19 Q11

Q39.

7 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

9231/11/M/J/20 Q7

5. Polar coordinates

Q40.

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$. [2]

9231/13/M/J/20 Q5

Q41.

7 **(a)** Show that the curve with Cartesian equation

$$(x^2 + y^2)^{\frac{5}{2}} = 4xy(x^2 - y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

(c) Find the exact value of the area of the region enclosed by C. [4]

(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

9231/11/O/N/20 Q7

Q42.

5 The curve C has polar equation $r = \ln(1 + \pi - \theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = 1 + \pi - \theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1 + \pi)\ln(1 + \pi)\big(\ln(1 + \pi) - 2\big) + \pi.$$ [6]

(c) Show that, at the point of C furthest from the initial line,

$$(1 + \pi - \theta)\ln(1 + \pi - \theta) - \tan\theta = 0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

9231/12/O/N/20 Q5

5. Polar coordinates

Q43.

5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

(c) Show that C has Cartesian equation $2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2 + y^2}$. [3]

9231/11/M/J/21 Q5

Q44.

5 The curve C has polar equation $r = \dfrac{1}{\pi - \theta} - \dfrac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\dfrac{3 - 4\ln 2}{4\pi}$. [6]

9231/13/M/J/21 Q5

Q45.

The curve C has polar equation $r = 2\cos\theta(1 + \sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

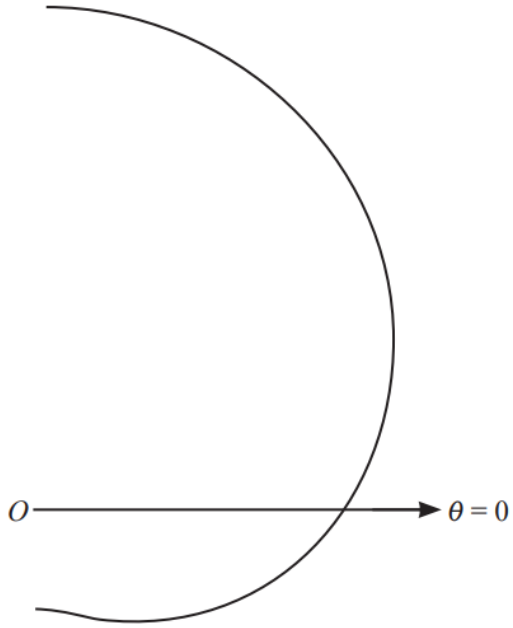
9231/11/O/N/21 Q6

5. Polar coordinates

Q46.

The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

9231/12/O/N/21 Q5

5. Polar coordinates

Q47.

The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

9231/11/M/J/22 Q6

Q48.

The curve C has Cartesian equation $x^2 + xy + y^2 = a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2 = \dfrac{2a}{2 + \sin 2\theta}$. [3]

(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1 + \tan^2\theta}$. Use this result to show that the area of R is

$$\tfrac{1}{2}a\int_0^{\frac{1}{4}\pi} \frac{1 + \tan^2\theta}{1 + \tan\theta + \tan^2\theta}\,\mathrm{d}\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

9231/13/M/J/22 Q6

5. Polar coordinates

Q49.

(a) Show that the curve with Cartesian equation

$$\left(x^2+y^2\right)^2 = 36\left(x^2-y^2\right)$$

has polar equation $r^2 = 36\cos 2\theta$. [3]

The curve C has polar equation $r^2 = 36\cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line, giving the answer in exact form. [6]

9231/11/O/N/22 Q6

Q50.

The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]

(b) Find the maximum distance of a point of C from the initial line. [2]

(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

(d) Find, in the form $y = \mathrm{f}(x)$, the Cartesian equation of C. [3]

9231/12/O/N/22 Q5

Solutions

Q1.

4 **(i)** $\theta = 2, r = 4$ B1B1

Ignore extra values

Accept as written in MS – co-ords not required

(ii) Graphs: correct location, orientation and concavity required B1B1

Separate diagrams 1/2, B1 Shapes correct

B1 Intersection correct

(iii) $A_1 = (1/2)\ \int_0^2 (\theta + 2)^2\, d\theta$ (LNR) M1

$= \ldots = 28/3$

$A_2 = (1/2)\ \int_0^2 \theta^4 d\theta\ = \ldots = 16/5$ (LR) A1

M1 for 1 correct integral representation plus A1 if both correct

Area $= A_1 - A_2 = 92/15$ (6.13) A1

S.C. –92/15 M1 A0 A1

Alternative layout: $\frac{1}{2}\int_0^2 (\theta + 2)^2 - \theta^4\, d\theta$ M1 (LNR)

$= \frac{1}{2}\left[\frac{\theta^3}{3} + 2\theta^2 + 4\theta - \frac{\theta^5}{5}\right]_0^2$ A1 (LR)

$= 92/15$ A1

5. Polar coordinates

Q2.

3	Approximately correct curve passing through the pole, O, and the point $A(\pi^2/4,0)$.	B1
	Negative gradient at A	B1
	Correct form at O.	B1
	$\text{Area} = (1/2)\int_0^{\pi/2} (\pi/2-\theta)^4\, d\theta$	M1
	$= -(1/10)\left[(\pi/2-\theta)^5\right]_0^{\pi/2}$	A1
	$= \pi^5/320$	A1

Q3.

5	Sketch with correct shape, location and orientation	B1
	Shows tangency to the initial line at the pole	B1
	Ignore extra in diagram	
	Draws, in the same diagram, a straight line passing through the origin and with positive gradient (distinct half-line and not a construction line)	B1
	$(1/2)\int_0^{\pi/2} \theta^2 d\theta = \pi^3/48$	M1A1
	$(1/2)\int_0^{\alpha} \theta^2 d\theta = \alpha^3/6$	A1
	$\alpha^3/6 = \pi^3/96 \Rightarrow \alpha = \pi.2^{-4/3}$ (aef)	A1
	or for previous 4 marks:	
	$(1/2)\int_0^{\alpha} \theta^2 d\theta = (1/2)\int_{\alpha}^{\pi/2} \theta^2 d\theta$	M1
	$\alpha^3/6 = (1/6)[(\pi/2)^3 - \alpha^3]$	A1A1
	$\Rightarrow \alpha^3 = \pi^3/16 \Rightarrow \alpha = \pi(16)^{-1/3}$, or equivalent	A1

5. Polar coordinates

Q4.

10 **(i)** Area $= \left(a^2/2\right)\int_0^{\pi/3} \sin^2 3\theta \, d\theta$ M1

$= \left(a^2/4\right)\int_0^{\pi/3} \left(1-\cos 6\theta\right) d\theta$ A1

$= \ldots = \pi a^2/12$ (AG) A1

(ii) Considers $y = a\sin 3\theta \sin\theta$ B1

$dy/d\theta = 0 \Rightarrow 3\cos 3\theta \sin\theta + \sin 3\theta \cos\theta = 0$ M1

$\Rightarrow \ldots \Rightarrow \tan 3\theta + 3\tan\theta = 0$ (AG) A1

(iii) Uses $\tan 3\theta = \left(3\tan\theta - \tan^3\theta\right)/\left(1-3\tan^2\theta\right)$ to obtain $\tan^2\theta = 3/5$ M1A1

Uses $y = \sin 3\theta \sin\theta$ with $\tan\theta = \sqrt{3/5}$ M1

Obtains $y = 9a/16$ (Accept $0.5625a$, or $0.563a$) A1

(iv) Closed loop entirely in the first quadrant with lower end at the pole and clearly tangential to the initial line at the pole B1

Symmetric about line $\theta = \pi/6$ with correct shape at $(a,\ \pi/6)$ B1

Q5.

2 **(i)** Sketch of C:

Approximately correct shape and location for $0 \le \theta < 2\pi$ B1

Shows initial line to be tangential to C at the pole B1

Asymptotic approach to circle $r = a$ B1

[3]

(ii) $A = (a^2/2)\int_{\ln 2}^{\ln 4} (1 - 2e^{-\theta} + e^{-2\theta}) d\theta$ M1A1

$= (a^2/2)[\theta + 2e^{-\theta} - (1/2)e^{-2\theta}]_{\ln 2}^{\ln 4}$ A1

$= \ldots = (a^2/2)(\ln 2 - 13/32)$ (AG) A1

[4]

5. Polar coordinates

Q6.

11 **(i)** $dr/d\theta = -a/(1+\theta)^2$ M1

Since $a > 0$, then $dr/d\theta < 0$, $\forall$ points of C A1
[2]

OR $a(>0)$ constant and as $\theta(>0)$ increases, $1+\theta$ increases M1

$\therefore \dfrac{a}{1+\theta}$ decreases A1

(ii) $y = a\sin\theta/(1+\theta)$ M1

$dy/d\theta = 0 \Rightarrow \ldots \Rightarrow (1+\theta)\cos\theta - \sin\theta = 0$ A1

$\Rightarrow \tan\theta = 1+\theta$ (AG) A1

$\tan\theta - 1 - \theta = -0.135$ when $\theta = 1.1$: $\tan\theta - 1 - \theta = +0.372$ when $\theta = 1.2$ B1
[4]

OR equivalent argument for B1 such as:

$\tan(1.1) \approx 1.96 < 2.1$, $\tan(1.2) \approx 2.37 > 2.2$ B1

or $f(\theta) = \tan\theta - (1+\theta) \Rightarrow f(1.1) = -0.14$ $f(1.2) = 0.37$

(iii) Sketch:

Approximately correct shape and placement for $0 \leqslant \theta \leqslant \pi/2$ and passing through $(a, 0)$ and $(0.4a, \pi/2)$, approximately, indicated in some way B1B1

Maximum in interval $(\pi/4, \pi/2)$ B1
[3]

(iv) $A = (a^2/2)\int_0^{\pi/2} (1+\theta)^{-2}\, d\theta$ M1

$= (a^2/2)\left[-(1+\theta)^{-1}\right]_0^{\pi/2}$ A1

$= \ldots = \pi a^2/2(\pi+2)$ A1

Do not accept double minus signs or fractions in the denominator for the final mark. [3]

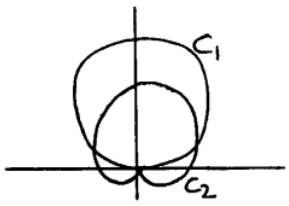
5. Polar coordinates

Q7.

8 **(i)** $1+\sin\theta = 3\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$ M1

$\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$ A1 (both) [2]

(ii)



B1 circle
B1 cardioid behaviour at origin
B1 cardioid closed and symmetry
[3]

(iii) Subtract integrands M1

$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2}(3-4\cos 2\theta - 2\sin\theta)\,d\theta$ M1

$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2}$ M1A1

$= \pi$ **AG** A1 [5]

Alternative:
Area inside C_1:

$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta = \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$ M1

$= \frac{9}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$ A1

Area inside C_2:

$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2} 1 + 2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \left[\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2}$ M1

$= \left(\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right)$ (A1 if not earned earlier)

Subtraction M1
Required area = π **AG** A1 [5]

Area inside C_2:

$$2\times\frac{1}{2}\int_{\pi/6}^{\pi/2}1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\,\mathrm{d}\theta$$

$$=\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{\pi/6}^{\pi/2} \qquad \text{M1}$$

$$=\left(\frac{\pi}{2}+\frac{9\sqrt{3}}{8}\right) \qquad \text{(A1 if not earned earlier)}$$

Subtraction M1

Required area = π **AG** A1 [5]

Q8.

5	Right-hand loop Left-hand loop Deduct 1 mark for extra loops ($r < 0$).	Position and through pole and (2, 0). Position and through pole and (2, π).	B1B1 B1B1	4	
	Uses $A = \frac{1}{2}\int r^2\,\mathrm{d}\theta$	$A = \frac{1}{2}\int 4\cos^2\theta\,\mathrm{d}\theta\,\mathrm{d}\theta$	M1		
	Uses double angle formula.	$=\int(\cos 4\theta+1)\,\mathrm{d}\theta$ (LNR)	M1		
	Integrates.	$=\left[\sin 4\theta+\theta\right]$ (LNR)	A1		
	Inserts *any appropriate* limits which legitimately give the result.	e.g. $\left[\sin 4\theta+\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{2}$	A1	4	**[8]**

Q9.

6	Sketches each curve on same diagram.	Sketch of C_1 (relevant part only required). Sketch of C_2 (generous on tangency features).	B1 B2	3	
	States the value of β.	$\beta=\frac{\pi}{6}$.	B1	1	
	Adds $\frac{1}{12}$ of area of circle to sector of C_2 from $\theta=\frac{\pi}{6}$ to $\theta=\frac{\pi}{2}$.	$\frac{1}{12}\pi a^2+\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}4a^2\cos^2 2\theta\,\mathrm{d}\theta$	B1M1		
	Uses double angle formula and integrates.	$=\frac{1}{12}\pi a^2+a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(\cos 4\theta+1)\,\mathrm{d}\theta$ $=\frac{1}{12}\pi a^2+a^2\left[\frac{\sin 4\theta}{4}+\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$	M1		
	Obtains printed result.	$=a^2\left(\frac{\pi}{6}-\frac{\sqrt{3}}{8}\right)$ (AG)	A1	4	**[8]**

5. Polar coordinates

Q10.

10	C: Straight line	Closed loop through (5,0) and $(1,\pi)$ Correct shape near $(1,\pi)$ Perpendicular to initial line , through (2,0)	B1 B1 B1	3	
	Forms quadratic equation in usual form.	$\Rightarrow (3+2\cos\theta)\cos\theta = 2$ $\Rightarrow 2\cos^2\theta + 3\cos\theta - 2 = 0$ (aef) $\Rightarrow (2\cos\theta - 1)(\cos\theta + 2) = 0$	M1		
	Solves quadratic equation.	$\Rightarrow \cos\theta = 0.5$ (since $\cos\theta > 0$)	A1		
	Writes down points of intersection.	Intersections at $\left(4, \frac{\pi}{3}\right)$ and $\left(4, -\frac{\pi}{3}\right)$.	A1A1	4	
		Calling points of intersection A and B and the pole O. Required area is two congruent sectors between l and C plus triangle OAB.			
	Finds required area.	Two sectors $= 2\times\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}(9+12\cos\theta+4\cos^2\theta)d\theta$	M1		
		$= \int_{\frac{\pi}{3}}^{\pi}(11+12\cos\theta+2\cos 2\theta)d\theta$	A1		
		$= \left[11\theta+12\sin\theta+\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$	M1		
		$= \frac{22\pi}{3} - \frac{13\sqrt{3}}{2} = (11.78)$	A1		
		Triangle $= 2\sqrt{3}\times 2 = 4\sqrt{3} = (6.928)$	B1		
		Total Area $= \frac{22\pi}{3} - \frac{5\sqrt{3}}{2} = (18.708 = 18.7\ (3sf))$	A1	6	**[13]**

5. Polar coordinates

Q11.

8	Sketches graph.	Arc above initial line. Arc below initial line.	B1 B1	2	
	Uses $\frac{1}{2}\int r^2 d\theta$	$\frac{1}{2}\int(1+\sin\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$	M1		
	Uses double angle formula.	$= \frac{1}{2}\int\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1		
	Integrates.	$= \frac{1}{2}\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]+c$	M1A1		
	Inserts limits.	$A_1 = \left[\frac{1}{2}\left(\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{8}+1$	M1A1		
		$A_2 = \left[\frac{1}{2}\left(\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right)\right]_{-\frac{\pi}{2}}^{0} = \frac{3\pi}{8}-1$	A1		
		$n = \left(\frac{3\pi}{8}+1\right)\div\left(\frac{3\pi}{8}-1\right) = 12.2$ (1d.p.)	A1	8	**[10]**

Q12.

4	Draws sketch of C.	Shows (4,0) and $(0,\pi)$ lie on C. Correct shape. (Full cardioid is B1 unless clear evidence of plotting up to 2π or $-\pi$ to π.)	B1 B1	2	
	Uses $\frac{1}{2}\int_\alpha^\beta r^2\mathrm{d}\theta$	$\frac{1}{2}\int_0^\pi(4+8\cos\theta+4\cos^2\theta)\mathrm{d}\theta$	M1		
	Uses double angle formula.	$= \int_0^\pi(2+4\cos\theta+2\cos^2\theta)\mathrm{d}\theta$			
		$= \int_0^\pi(3+4\cos\theta+\cos 2\theta)\mathrm{d}\theta$	M1		
	Integrates and obtains area.	$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^\pi = 3\pi$ (A1 for **correct** integral)	A1A1 **CWO**	4	
	Finds areas.	$\frac{3\pi}{5}+4\sin\frac{\pi}{5}+\sin\frac{\pi}{5}\cos\frac{\pi}{5} = 4.712$	M1A1		
		$3\pi - 4.712 = 4.713$	A1	3	**[9]**

5. Polar coordinates

Q13.

11	**EITHER**				
	Uses $x^2+y^2=r^2$, $x=r\cos\theta$ and $y=r\sin\theta$.	$(x^2+y^2)^2=a^2(x^2-y^2)$ $\Rightarrow r^2=a^2\left(\frac{x^2}{r^2}-\frac{y^2}{r^2}\right)$ $=a^2(\cos^2\theta-\sin^2\theta)=a^2\cos 2\theta$ (AG)	M1 A1	 2	
	One mark for each loop, or half of whole curve.	Sketches C.	B2,1,0	2	
	Uses sector area formula.	$\text{Area}=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos 2\theta\,\mathrm{d}\theta\left(=\int_0^{\frac{\pi}{4}}a^2\cos 2\theta\,\mathrm{d}\theta\right)$	M1		
	S.C. Omission of ½ factor, but correct integration gets B1.	$=a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}=\frac{a^2}{2}$	A1A1	3	
	Differentiates. Puts $y'=0$. Obtains coordinates.	$2(x^2+y^2)(2x+2yy')=a^2(2x-2yy')$ $y'=0\Rightarrow 2x(x^2+y^2)=a^2x$ $\Rightarrow 2r^2=a^2\Rightarrow r=\frac{a}{\sqrt{2}}\quad(r\geq 0)$	B1B1 M1 A1		
		$\Rightarrow\cos 2\theta=\frac{1}{2}$	M1		
		$\Rightarrow\theta=\pm\frac{\pi}{6},\pm\frac{5\pi}{6}$	A1A1	7	
		i.e. $\left(\frac{a}{\sqrt{2}},\pm\frac{\pi}{6}\right)$ and $\left(\frac{a}{\sqrt{2}},\pm\frac{5\pi}{6}\right)$			**[14]**

Q14.

1	Re-writes equation	$r\cos\left(\theta-\frac{\pi}{4}\right)=\sqrt{2}$	M1		
	and uses compound angle formula.	$r\left(\frac{\cos\theta}{\sqrt{2}}+\frac{\sin\theta}{\sqrt{2}}\right)=\sqrt{2}$	A1		
	Changes to cartesian.	$\Rightarrow r\cos\theta+r\sin\theta=2\Rightarrow x+y=2$ or $y=2-x$.	A1	3	
	Sketches graph.	Straight line at $-\frac{\pi}{4}$ to the initial line.	B1		
		Point (2,0) clearly indicated.	B1	2	**[5]**

5. Polar coordinates

Q15.

5	Sketches graph.	Correct shape and orientation. Passing through (0,0) and (3,0)	B1 B1	2	
	Uses area of sector formula.	Area $= 2\times\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+2\cos\theta)^2\,d\theta$	M1		
	Uses double angle formula Integrates.	$=\int_0^{\frac{\pi}{3}}(1+4\cos\theta+4\cos^2\theta)d\theta$ $=\int_0^{\frac{\pi}{3}}(3+4\cos\theta+2\cos 2\theta)\,d\theta$	M1		
		$=\left[3\theta+4\sin\theta+\sin 2\theta\right]_0^{\frac{\pi}{3}}$	A1		
	Obtains result.	$=\left[\pi+\frac{5}{2}\sqrt{3}\right]$	A1	4	**[6]**

Q16.

1	Use of $\frac{1}{2}\int r^2\,d\theta$	$A=\frac{1}{2}\int_0^{2\pi}4(1+2\cos\theta+\cos^2\theta)\,d\theta$	M1		
	Use of double angle formula and attempt to integrate.	$=\int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta$	M1		
	Integrates correctly.	$=\left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}$	A1		
	Finds value.	$=6\pi$ (CWO) Accept 18.8	A1	4	**[4]**

Q17.

10	Uses identity.	$2\sin\theta\cos\left(\theta-\frac{1}{4}\pi\right)=\sin\left(2\theta-\frac{1}{4}\pi\right)+\sin\left(\frac{1}{4}\pi\right)$	M1		
	Uses sine-cosine link. Obtains result.	$=\cos\left(\frac{1}{2}\pi-2\theta+\frac{1}{4}\pi\right)+\frac{1}{\sqrt{2}}$ $=\cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{\sqrt{2}}$ (AG)	M1 A1	3	
	Sketches graph.	Closed loop through origin, in correct position.	B1		
	Obtains line of symmetry.	For line of symmetry $2\theta-\frac{3}{4}\pi=0\Rightarrow\theta=\frac{3}{8}\pi$.	B1B1	3	
	Uses area of sector formula. Rearranges.	$A=\frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\cos^2\left(2\theta-\frac{3}{4}\pi\right)+\sqrt{2}\cos\left(2\theta-\frac{3}{4}\pi\right)+\frac{1}{2}\right\}d\theta$ $=\frac{1}{2}\int_0^{\frac{3}{4}\pi}\left\{\frac{1}{2}\cos\left[4\theta-\frac{3}{2}\pi\right]+\sqrt{2}\cos\left[2\theta-\frac{3}{4}\pi\right]+1\right\}d\theta$	M1 A1		
	Integrates correctly.	$=\left[\frac{1}{16}\sin\left(4\theta-\frac{3}{2}\pi\right)+\frac{1}{2\sqrt{2}}\sin\left(2\theta-\frac{3}{4}\pi\right)+\frac{\theta}{2}\right]_0^{\frac{3}{4}\pi}$	dM1A1		
	Substitutes limits.	$=\left[-\frac{1}{16}+\frac{1}{4}+\frac{3}{8}\pi\right]-\left[\frac{1}{16}-\frac{1}{4}\right]$	dM1		
	Obtains given answer.	$=\frac{3}{8}(\pi+1)$ (AG)	A1	6	**[12]**
		N.B Method marks are dependent in final part. If $\frac{1}{2}$ factor missing throughout – award M's (Max 3) If $2\times\frac{1}{2}\int_0^{\frac{3}{8}}r^2d\theta$, penultimate line is $=\left[\frac{3}{8}\pi\right]-\left[\frac{1}{8}-\frac{1}{2}\right]$			

5. Polar coordinates

Q18.

1 (i)	Uses area formula.	$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4e^{2\theta}\,d\theta = \left[e^{2\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$	M1		
	Obtains result.	$= e^{\pi} - e^{\frac{\pi}{3}}$ $(= 20.3)$	A1	2	
(ii)	Uses arc length formula.	$\text{Arc length} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4e^{2\theta} + 4e^{2\theta}}\,d\theta = 2\sqrt{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} e^{\theta}\,d\theta$	M1A1		
	Obtains result.	$= 2\sqrt{2}\left[e^{\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 2\sqrt{2}\left[e^{\frac{\pi}{2}} - e^{\frac{\pi}{6}}\right]$ $(= 8.83)$	A1	3	**[5]**

Q19.

10	Differentiates wrt θ	$\frac{dr}{d\theta} = 2\cos\theta - 2\cos 2\theta$	M1		
	Equates to zero	$2c - 2(2c^2 - 1) = 0 \Rightarrow 2c^2 - c - 1 = 0$	A1		
	And solves equation	$\Rightarrow (2c+1)(c-1) = 0$	M1		
		$\Rightarrow c = -\frac{1}{2}$ or 1	A1		
	States required points on C	$\left(\frac{3}{2}\sqrt{3}, \frac{2}{3}\pi\right)$	A1	**(5)**	
	Sketches C.	Approximate shape and location Accurate scaling.	B1 B1	**(2)**	
	Uses $\frac{1}{2}\int r^2 d\theta$	$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2\theta(1 - 2\cos\theta + \cos^2\theta)\,d\theta$	M1 M1		
	Obtains an integrable form	$= \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta - 4\cos\theta\sin^2\theta + \frac{1}{4}[1 - \cos 4\theta]\right) d\theta$	A1A1		
	Integrates	$= \left[\frac{5\theta}{4} - \frac{\sin 2\theta}{2} - 4\frac{\sin^3\theta}{3} - \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{4}}$	M1		
	Obtains result	$= \frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$ or 0.0103	A1	**(6)**	**[13]**

5. Polar coordinates

Q20.

5	Closed curve through $(a, 0)$, $\left(2a, \frac{1}{2}\pi\right)$, (a, π), $\left(0, \frac{3}{2}\pi\right)$. Completely correct shape for cardioid. $\text{Area} = \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+\sin\theta)^2\,\mathrm{d}\theta = \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+2\sin\theta+\sin^2\theta)\mathrm{d}\theta$ (LNR) $= \frac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ (LR) $= \frac{a^2}{2}\left[\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}$ $= a^2\left(\frac{1}{4}\pi+1+\frac{1}{8}\sqrt{3}\right)$ (CAO)	B1 B1 [2] M1 M1 M1 A1 [4]

Q21.

4	Use of $r^2 = x^2 + y^2$ Use of $x = r\cos\theta$ and $y = r\sin\theta$ (both). Obtains $r^2 = a^2\sin 2\theta$ (AG)	B1 B1 B1 [3]
	Sketch with two loops, approximately symmetrical about $\theta = \frac{1}{4}\pi$ and $\theta = -\frac{3}{4}\pi$.	B1B1 [2]
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} a^2\sin 2\theta\,\mathrm{d}\theta = \left[-\frac{a^2}{4}\cos 2\theta\right]_0^{\frac{1}{2}\pi}$ (LNR) $= \frac{1}{2}a^2$	M1 A1 (2)

5. Polar coordinates

Q22.

8	Circle sketched Cardioid – correct location and orientation – correct indentation near pole. $(a,\frac{\pi}{2})$ and $(a,\frac{3\pi}{2})$ (B1 for reverse, or $a=a(1-\cos\theta)\Rightarrow\theta=\frac{\pi}{2},\frac{3\pi}{2}$ seen.) Area $=\frac{1}{2}\pi a^2+2\times\frac{1}{2}\int_0^{\frac{\pi}{2}}a^2(1-\cos\theta)^2\,\mathrm{d}\theta$ (Half circle + Area of sector) $=\frac{1}{2}\pi a^2+a^2\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\cos^2\theta\right)\mathrm{d}\theta$ $=\frac{1}{2}\pi a^2+a^2\int_0^{\frac{\pi}{2}}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ (Use of double angle formula.) $=\frac{1}{2}\pi a^2+a^2\left[\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ (Integration) $=\frac{1}{2}\pi a^2+a^2\left(\frac{3}{4}\pi-2\right)=\left(\frac{5}{4}\pi-2\right)a^2$ (AG)	B1 B1B1 (3) B1B1 (2) B1M1 A1 M1 M1 A1 (6) **[11]**

Q23.

5	$\sin\frac{1}{2}\theta=\frac{1}{2}\Rightarrow\theta=\frac{1}{3}\pi$; Intersection at $\left(\frac{1}{\sqrt{2}},\frac{1}{3}\pi\right)$ (Accept 1.05 for $\frac{1}{3}\pi$.) C_1: Circle centre at pole and radius $1/\sqrt{2}$ C_2: Curve, approx. correct orientation, from (0,0) to $(1,\pi)$. Completely correct correct shape. $\frac{1}{6}\times\pi\times\frac{1}{2}-\frac{1}{2}\int_0^{\frac{1}{3}\pi}\sin\frac{1}{2}\theta\,\mathrm{d}\theta$ $=\frac{1}{12}\pi\times\frac{1}{2}\left[-2\cos\frac{1}{2}\theta\right]_0^{\frac{1}{3}\pi}=\frac{1}{12}\pi+\frac{\sqrt{3}}{2}-1$	M1A1 (2) B1 B1 B1 (3) M1A1 M1A1 (4) **Total: 9**

Q24.

2	$s=\int_0^{\alpha}\sqrt{\mathrm{e}^{8\theta}+16\mathrm{e}^{8\theta}}\,\mathrm{d}\theta=\int_0^{\alpha}\sqrt{17}\mathrm{e}^{4\theta}\mathrm{d}\theta$ = (AEF) (LNR) $=\left[\frac{\sqrt{17}}{4}\mathrm{e}^{4\theta}\right]_0^{\alpha}\Rightarrow\frac{\sqrt{17}}{4}\mathrm{e}^{4\alpha}-\frac{\sqrt{17}}{4}=2015$ $\Rightarrow\mathrm{e}^{4\alpha}=1955.837\ \ldots\Rightarrow\alpha=1.89$ CAO	M1A1 dM1A1 dM1A1 (6) **Total 6**

5. Polar coordinates

Q25.

11O	Closed curve through pole with correct orientation. Completely correct. $2\times\frac{1}{2}a^2\int_{\frac{1}{2}\pi}^{\pi}(1-2\cos\theta+\cos^2\theta)\mathrm{d}\theta = a^2\int_{\frac{1}{2}\pi}^{\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ $= a^2\left[\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_{\frac{1}{2}\pi}^{\pi}$ $= a^2\left(\frac{3}{4}\pi+2\right)$ $\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)$ $= 2a^2(1-\cos\theta) = 2a^2.2\sin^2\frac{1}{2}\theta = 4a^2\sin^2\frac{1}{2}\theta$ (AG) $s = 2\times\int_{\frac{1}{2}\pi}^{\pi}2a\sin\frac{1}{2}\theta\,\mathrm{d}\theta$ $= 4a\left[-2\cos\frac{1}{2}\theta\right]_{\frac{1}{2}\pi}^{\pi}$ $= 4\sqrt{2}a$	**B1** **B1** [2] **M1M1** **M1A1** **A1** [5] **B1** **M1A1** **M1** **A1** **M1A1** [7] **Total 14**

Q26.

4	Using $x = r\cos\theta$ **and** $y = r\sin\theta$ $r^2 = 8\,\mathrm{cosec}\,2\theta \Rightarrow r^2 = \frac{4}{\sin\theta\cos\theta}$ $\Rightarrow r\cos\theta.r\sin\theta = 4 \Rightarrow xy = 4$ (in simple form) Sketch: Curve in 1st quadrant with correct concavity, asymptotic to **both** axes. $\frac{1}{2}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}8\,\mathrm{cosec}\,2\theta\,\mathrm{d}\theta = \left[2\ln|\tan\theta|\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$ $= 2\left\{\ln\left|\sqrt{3}\right| - \ln\left|\frac{1}{\sqrt{3}}\right|\right\} = 2\ln 3 \text{ or } \ln 9$	**B1** **M1** **A1** [3] **B1B1** [2] **M1A1** **A1** [3]

5. Polar coordinates

Q27.

7	$\sqrt{x^2+y^2}=\dfrac{1}{1-\dfrac{x}{\sqrt{x^2+y^2}}} \Rightarrow \sqrt{x^2+y^2}-x=1$	**M1A1**
	$\Rightarrow y^2=1+2x$ or equivalent RHS	**A1** [3]
	Sketch: Parabola symmetrical about x-axis. Intercepts at $\left(-\frac{1}{2},0\right)$ and $(0,\pm 1)$.	**B1 B1**
	Shading correct area	**B1** [3]
	Recognises $\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{(1-\cos\theta)^2}\,d\theta$ as the area of the region between parabola and y-axis.	
	$=2\times\int_{-\frac{1}{2}}^{0}(1+2x)^{\frac{1}{2}}\,dx$	**M1**
	$=2\left[\frac{1}{3}(1+2x)^{\frac{3}{2}}\right]_{-\frac{1}{2}}^{0}=\frac{2}{3}$	**M1A1** [4]

Q28.

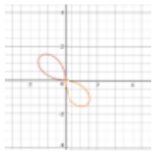
11 O	$\dot{x}^2+\dot{y}^2=36t^2+1-18t^2+81t^4=1+18t^2+81t^4=\left(1+9t^2\right)^2 \quad (\text{AG})$	**M1A1**	[2]
(i)	$s=\int_0^{\frac{1}{\sqrt{3}}}\left(1+9t^2\right)\mathrm{d}t=\left[t+3t^3\right]_0^{\frac{1}{\sqrt{3}}}=\frac{2}{\sqrt{3}}$	**M1A1**	[2]
(ii)	$S=2\pi\int_0^{\frac{1}{\sqrt{3}}}t\left(1-3t^2\right)\left(1+9t^2\right)\mathrm{d}t=2\pi\int_0^{\frac{1}{\sqrt{3}}}\left(t+6t^3-27t^5\right)\mathrm{d}t$	***M1**	
	$=2\pi\left[\frac{1}{2}t^2+\frac{3}{2}t^4-\frac{9}{2}t^6\right]_0^{\frac{1}{\sqrt{3}}}=\frac{\pi}{3}$	**DM1A1**	[3]
	Substitute $t=\frac{y}{x}$ in one of the parametric equations to obtain e.g. $x=1-3\frac{y^2}{x^2}$.	**M1**	
	Uses $x=r\cos\theta$ and $y=r\sin\theta \Rightarrow r\cos\theta=1-3\tan^2\theta \Rightarrow r=\sec\theta\left(1-3\tan^2\theta\right)$.	**M1A1**	
	States domain for θ : $0\leqslant\theta\leqslant\frac{\pi}{6}$,	**B1**	[4]
	$A=\frac{1}{2}\int_0^{\frac{\pi}{6}}\sec^2\theta\left(1-6\tan^2\theta+9\tan^4\theta\right)\mathrm{d}\theta$	**M1**	
	$=\frac{1}{2}\left[\tan\theta-2\tan^3\theta+\frac{9}{5}\tan^5\theta\right]_0^{\frac{\pi}{6}}=\frac{4}{45}\sqrt{3}$	**A1A1**	[3]

5. Polar coordinates

Q29.

11(ii)	$A=\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$	**M1**	
	$=\left(\frac{a^2}{2}\right)\int_{-\pi}^{\pi}\left(1+2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	**M1**	
	$=\left(\frac{a^2}{2}\right)\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{-\pi}^{\pi}=\frac{3\pi a^2}{2}$	**M1A1**	
	Total:	**4**	
11(iii)	Show that when r=0, $\theta=-\frac{\pi}{2}$, when r=2a $\theta=\frac{\pi}{2}$ and that $\frac{\mathrm{d}r}{\mathrm{d}\theta}=a\cos\theta$	**B1**	
	$s=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2\left(1+2\sin\theta+a^2\sin^2\theta\right)+a^2\cos^2\theta}\ \mathrm{d}\theta=\sqrt{2a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{1+\sin\theta}\ \mathrm{d}\theta$	**M1A1**	Uses correct formula for arc length; AG
	Total:	**3**	
11(iv)	$u=1+\sin\theta\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}\theta}=\cos\theta=\sqrt{1-(u-1)^2}=\sqrt{2u-u^2}$	**M1**	
	$so\ s=\sqrt{2a}\int_0^2\frac{1}{\sqrt{2-u}}\mathrm{d}u$ AG	**A1**	Including limits
	$=\sqrt{2a}\left[-2(2-u)^{\frac{1}{2}}\right]_0^2=4a$	**M1A1**	
	Total:	**4**	

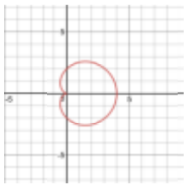
Q30.

11E(i)	$r=2a\left(\cos 2\theta\cos\frac{1}{2}\pi-\sin 2\theta\sin\frac{1}{2}\pi\right)=2a(0-\sin 2\theta.1)=-2a\sin 2\theta$	**M1A1**	AG
		**B1**	Loops in 2nd and 4th quadrant
		B1	Symmetry and shape correct
	Total:	**4**	
11E(ii)	$r=-4a\sin\theta\cos\theta=-4a.\frac{y}{r}.\frac{x}{r}\Rightarrow r^3=-4axy\Rightarrow\left(\left(x^2+y^2\right)^{\frac{3}{2}}=-4axy\right)$	**M1A1**	AG
	Total:	**2**	
11E(iii)	Area of one loop is $\frac{1}{2}\int_{\frac{1}{2}\pi}^{\pi}4a^2\sin^2 2\theta\mathrm{d}\theta=a^2\int_{\frac{1}{2}\pi}^{\pi}2\sin^2 2\theta\mathrm{d}\theta$	**M1**	OE
	$=a^2\int_{\frac{1}{2}\pi}^{\pi}(1-\cos 4\theta)\mathrm{d}\theta$	**M1A1**	
	$=a^2\left[\theta-\frac{1}{4}\sin 4\theta\right]_{\frac{1}{2}\pi}^{\pi}\quad=\frac{1}{2}\pi a^2$	**M1A1**	
	Total:	**5**	

5. Polar coordinates

11E(iv)	$y = r\sin\theta = -2a\sin 2\theta\sin\theta$	**B1**	
	$\frac{dy}{d\theta} = -2a\left(2\cos 2\theta\sin\theta + \sin 2\theta\cos\theta\right) = 0$	**M1**	
	$2\cos 2\theta\sin\theta = -\sin 2\theta\cos\theta \quad (\Rightarrow 2\tan\theta = -\tan 2\theta)$	**A1**	AG
	Total:	**3**	

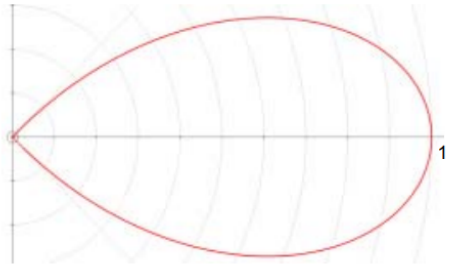
Q31.

11OR		**B1**	Closed curve starting and ending at pole, in approximately correct location.
		B1	Cardioid with indication of correct scale.
		2	
11OR(ii)	$r = a(1+\cos\theta) \Rightarrow \sqrt{x^2+y^2} = a\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)$	**M1**	
	$x^2+y^2 = a(x+\sqrt{(x^2+y^2)})$	**A1**	Substitutes for r **and** cos(θ)
		2	

11OR(iii)	Sector area $= \frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$ $= \frac{a^2}{2}\int_0^{\frac{1}{3}\pi}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$	**M1A1**	
	$= \frac{a^2}{2}\left[\frac{3\theta}{2}+2\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\frac{1}{3}\pi}$	**M1**	
	$= \frac{a^2}{16}\left(4\pi+9\sqrt{3}\right)$	**A1**	
		4	
11OR(iv)	Arc length $= \int_0^{\frac{1}{3}\pi}\sqrt{a^2\left(1+2\cos\theta+\cos^2\theta\right)+a^2\left(-\sin\theta\right)^2}\,d\theta$	**M1A1**	
	$= a\int_0^{\frac{1}{3}\pi}\sqrt{2+2\cos\theta}\,d\theta$	**A1**	
	$= a\left[4\sin\frac{\theta}{2}\right]_0^{\frac{1}{3}\pi} = 2a$	**M1A1**	
		5	

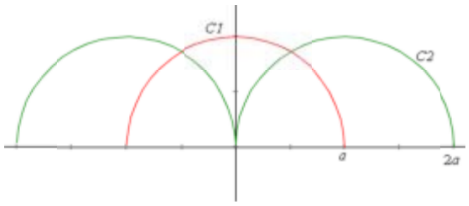
5. Polar coordinates

Q32.

3(i)		**B1**	Correct position including label of 1 on initial line, and symmetric about initial
		B1	Single correct loop.
		2	
3(ii)	$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} cos^2 2\theta d\theta$	**M1**	Correct integral
	$=\frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 4\theta + 1 d\theta =$	**M1**	Correct use of double angle formula
	$\frac{1}{4}\left[\frac{1}{4}\sin 4\theta + \theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8} = 0.393$	**A1**	
		3	

3(iii)	$r = \cos^2\theta - \sin^2\theta$ OE	**B1**	Uses trig identity
	Thus $\left(x^2+y^2\right)^{\frac{3}{2}} = x^2 - y^2$	**M1 A1**	Uses $x = r\cos\theta$ or $y = r\sin\theta$ *or both*, AEF
		3	

Q33.

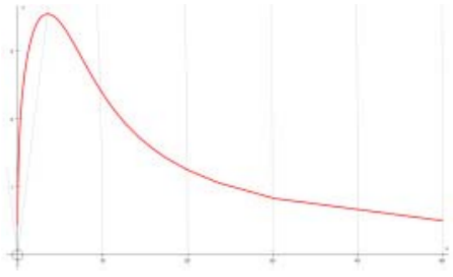
8(i)	$a = 2a\lvert\cos\theta\rvert \Rightarrow \cos\theta = \pm\frac{1}{2}$	**M1**	Eliminates r.
	$\left(a, \frac{\pi}{3}\right)$ and $\left(a, \frac{2\pi}{3}\right)$	**A1**	Both points needed for A1.
8(ii)		**B1**	Semicircle for C1 including r = a.
		B1	Half of C2 including r = 2a.
		B1	Other half of C2 and line of symmetry.

5. Polar coordinates

8(iii)	$4a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta$	**M1**	Finds area of segment OP_1 and OP_2 of C_2
	$=2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos 2\theta+1\,\mathrm{d}\theta$	**M1**	Uses $\cos^2\theta=\frac{1}{2}(\cos 2\theta+1)$
	$=2a^2\left[\frac{1}{2}\sin 2\theta+\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}=2a^2\left(\frac{\pi}{2}-\left(\frac{\sqrt{3}}{4}+\frac{\pi}{3}\right)\right)=a^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$	**A1**	Integrates correctly.
	Area $=\frac{\pi a^2}{6}-a^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)=-\frac{\pi a^2}{6}+\frac{a^2\sqrt{3}}{2}$	**M1** **A1FT**	M1 for subtracting 'their' OP_1P_2 from $\frac{\pi a^2}{6}$
		10	

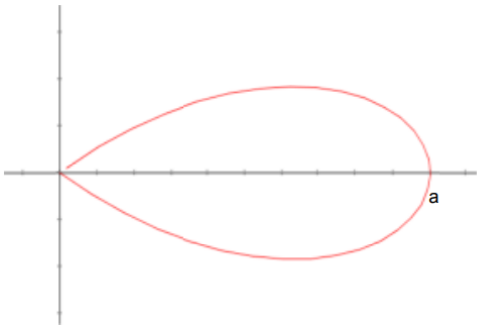
Q34.

9(i)	$\frac{25}{2}\int_{0.01}^{\frac{\pi}{2}}\cot\theta\,\mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$.
	$=\frac{25}{2}[\ln\sin\theta]_{0.01}^{\frac{\pi}{2}}$	**A1**	
	$=-\frac{25}{2}\ln\sin 0.01\approx 57.6$	**A1**	
		3	
9(ii)	$y=5\cos^{\frac{1}{2}}\theta\sin^{\frac{1}{2}}\theta=\frac{5}{\sqrt{2}}\sin^{\frac{1}{2}}2\theta$	**M1**	Uses $y=r\sin\theta$.
	$\theta=0.01\Rightarrow y\approx 0.5$	**A1**	
		2	
9(iii)	$\frac{\mathrm{d}y}{\mathrm{d}\theta}=\frac{5}{\sqrt{2}}\sin^{-\frac{1}{2}}2\theta\cos 2\theta=0$ or $\max(\sin 2\theta)=1$	**M1 A1**	Sets $\frac{\mathrm{d}y}{\mathrm{d}\theta}=0$ or considers max (AEF).
	$\Rightarrow y=\frac{5\sqrt{2}}{2}\ (=3.54)$	**A1**	
		3	

9(iv)		**B1**	Intersecting the initial line only when $x=0$ and $y=0$.
		B1	Correct shape.
		2	

5. Polar coordinates

Q35.

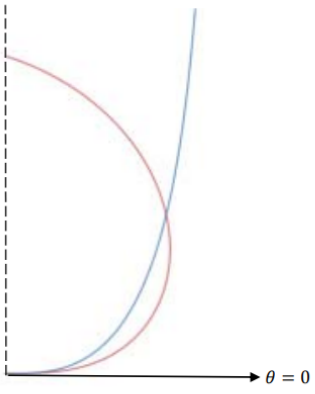
3(i)		**B1**	Just one loop, correct shape at extremities
		B1	Correct position including (a, 0) labelled or in table.
		2	

3(ii)	$\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2 cos^2 3\theta \, d\theta$	**M1**	For using correct formula
	$\frac{a^2}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 6\theta + 1) d\theta$	**M1**	Using double angle formula correctly
	$= \frac{a^2}{4}\left[\frac{1}{6}\sin 6\theta + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi a^2}{12}$	**A1**	
		3	
3(iii)	$r = a\cos\theta\left(4\cos^2\theta - 3\right) \Rightarrow r = a\left(\frac{x}{r}\right)\left(4\left(\frac{x}{r}\right)^2 - 3\right)$	**B1**	Uses $x = r\cos\theta$ **and** $x^2 + y^2 = r^2$.
	$\Rightarrow r^4 = ax\left(4x^2 - 3r^2\right) \Rightarrow \left(x^2 + y^2\right)^2 = ax\left(4x^2 - 3\left(x^2 + y^2\right)\right)$	**M1**	For eliminating θ
	$\Rightarrow \left(x^2 + y^2\right)^2 = ax\left(x^2 - 3y^2\right)$	**A1**	Any equivalent cartesian form without fractions.
		3	

Q36.

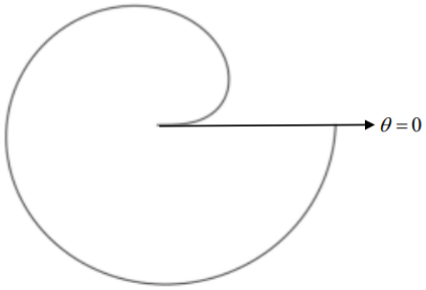
11E(i)	$x = \sqrt{2}\theta^{\frac{1}{2}}\cos\theta$	**M1**	Uses $x = r\cos\theta$.
	$\frac{d}{d\theta}\left(\sqrt{2}\theta^{\frac{1}{2}}\cos\theta\right) = \sqrt{2}\left(-\theta^{\frac{1}{2}}\sin\theta + \frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta\right) = 0$	**M1 A1**	Sets derivative of $r\cos\theta$ equal to zero.
	$-\theta^{\frac{1}{2}}\sin\theta + \frac{1}{2}\theta^{-\frac{1}{2}}\cos\theta = 0 \Rightarrow \cos\theta = 2\theta\sin\theta \Rightarrow 2\theta\tan\theta = 1$	**A1**	AG
	$2(0.6)\tan(0.6) - 1 = -0.179$ and $2(0.7)\tan(0.7) - 1 = 0.179$	**B1**	Shows sign change.
		5	

5. Polar coordinates

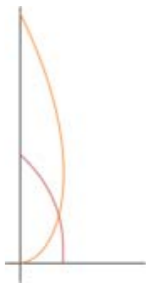
11E(ii)	$2\theta = \theta\sec^2\theta \Rightarrow \theta^{\frac{1}{2}}\left(\sec\theta - \sqrt{2}\right) = 0 \Rightarrow \theta = \frac{\pi}{4}$	**M1 A1**	Finds value of θ.
		2	
11E(iii)		**B1**	Correct shape.
		B1	Intersection correct.
		2	
11E(iv)	$\frac{1}{2}\int_0^{\frac{\pi}{4}} 2\theta\,\mathrm{d}\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} \theta\left(2 - \sec^2\theta\right)\mathrm{d}\theta$	**M1**	Forms correct integral.
	$\frac{1}{2}\left[\theta\left(2\theta - \tan\theta\right)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(2\theta - \tan\theta\right)\mathrm{d}\theta$	**M1 A1**	Integrates by parts.
	$\frac{1}{2}\left[\theta\left(2\theta - \tan\theta\right)\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[\theta^2 + \ln\cos\theta\right]_0^{\frac{\pi}{4}}$	**A1**	
	$\frac{\pi}{8}\left(\frac{\pi}{2} - 1\right) - \frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}\ln 2\right) = \frac{1}{4}\ln 2 + \frac{\pi}{8}\left(\frac{\pi}{4} - 1\right).$	**A1**	AEF, must be exact.
		5	

5. Polar coordinates

Q37.

2(i)		**B1**	Correct shape, domain and orientation.
		B1	The initial line is tangential to C at the pole.
		2	
2(ii)	$A = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\,\mathrm{d}\theta$	**M1**	States $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct expression and limits.
	$\int_0^{2\pi} \ln(1+\theta)\,\mathrm{d}\theta = \int_1^{2\pi+1} \ln u\ \mathrm{d}u$	**M1**	Applies given substitution correctly, changes their limits.
	$\int_1^{2\pi+1} \ln u\ \mathrm{d}u = \left[u\ln u\right]_1^{2\pi+1} - \left[u\right]_1^{2\pi+1}$	**M1 A1**	Integrates $\ln u$ (by parts or otherwise).
	$A = \left(\pi + \frac{1}{2}\right)\ln(2\pi+1) - \pi$	**A1**	AEF, must be exact.
		5	

Q38.

11O(i)	$\mathrm{e}^{2\alpha} - \mathrm{e}^{-2\alpha} = 2\left(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}\right) \Rightarrow \mathrm{e}^{\alpha} - \mathrm{e}^{-\alpha} = 2$	**M1**	Sets equations equal and divides by $\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}$.
	$\mathrm{e}^{2\alpha} - 2\mathrm{e}^{\alpha} - 1 = 0 \Rightarrow \mathrm{e}^{\alpha} = 1 + \sqrt{2}$	**M1 A1**	Forms quadratic in e^{α}, AG.
	$\alpha = \ln\left(1+\sqrt{2}\right)$	**A1**	Must be exact.
	$r = 2\left(1 + \sqrt{2} + \sqrt{2} - 1\right) = 4\sqrt{2}$	**M1 A1**	Substitutes to find r.
		6	
11O(ii)		**B1**	C_1 has correct shape.
		B1	C_2 has correct shape.
		B1	Intersection points positioned correctly.
		3	

5. Polar coordinates

11O(iii)	$2\int_0^{\ln(1+\sqrt{2})}\left(e^{\theta}+e^{-\theta}\right)^2\,d\theta-\frac{1}{2}\int_0^{\ln(1+\sqrt{2})}\left(e^{2\theta}-e^{-2\theta}\right)^2\,d\theta$ $=\int_0^{\ln(1+\sqrt{2})}5+2e^{2\theta}+2e^{-2\theta}-\frac{1}{2}e^{4\theta}-\frac{1}{2}e^{-4\theta}\,d\theta$	**M1 A1**	Uses $\frac{1}{2}\int r^2\,d\theta$ to formulate correct area.
	$=\left[5\theta+e^{2\theta}-e^{-2\theta}-\frac{1}{8}e^{4\theta}+\frac{1}{8}e^{-4\theta}\right]_0^{\ln(1+\sqrt{2})}$	**M1 A1**	Expands and integrates.
	$=5\ln(1+\sqrt{2})+\left(1+\sqrt{2}\right)^2-\left(1+\sqrt{2}\right)^{-2}-\frac{1}{8}\left(\left(1+\sqrt{2}\right)^4-\left(1+\sqrt{2}\right)^{-4}\right)=5.82$	**A1**	
		5	

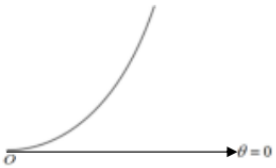
Q39.

7(a)	$x=\theta\cos^2\theta$	**B1**
	$\frac{dx}{d\theta}=-2\theta\cos\theta\sin\theta+\cos^2\theta=0$	**M1 A1**
	$\cos\theta\neq 0\Rightarrow 2\theta\tan\theta-1=0$	**A1**
	$2(0.6)\tan 0.6-1=-0.179$ and $2(0.7)\tan 0.7-1=0.179$	**B1**
		5
7(b)	$\theta\cos\theta=\theta\sin\theta\Rightarrow\tan\theta=1$	**M1**
	$\left(\frac{1}{8}\pi\sqrt{2},\frac{1}{4}\pi\right)$	**A1**
		2
7(c)	 (**B1** for initial line drawn and C_1 correct, **B1** for C_2 correct, **B1** for intersection correct)	**B1** **B1** **B1**
		3

5. Polar coordinates

7(d)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\left(\cos^2\theta-\sin^2\theta\right)\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\int_0^{\frac{1}{4}\pi}\theta^2\cos 2\theta\,\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\int_0^{\frac{1}{4}\pi}\theta\sin 2\theta\,\mathrm{d}\theta\right)$	**M1 A1**
	$=\frac{1}{2}\left(\left[\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\frac{1}{4}\pi}-\left[-\frac{1}{2}\theta\cos 2\theta\right]_0^{\frac{1}{4}\pi}-\frac{1}{2}\int_0^{\frac{1}{4}\pi}\cos 2\theta\,\mathrm{d}\theta\right)$	**M1**
	$=\frac{1}{4}\left[\theta^2\sin 2\theta+\theta\cos 2\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{1}{4}\pi}$	**A1**
	$=\frac{1}{64}\pi^2-\frac{1}{8}$	**A1**
		7

Q40.

5(a)		**B1**
	Maximum distance of C from the pole is a.	**B1**
		2
5(b)	$\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\tan^2\theta\,\mathrm{d}\theta$	**M1**
	$=\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^2\theta-1\,\mathrm{d}\theta=\frac{1}{2}a^2\left[\tan\theta-\theta\right]_0^{\frac{1}{4}\pi}$	**M1 A1**
	$=\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)$	**A1**
		4
5(c)	$\sqrt{x^2+y^2}=a\frac{y}{x}$	**B1**
	$x^2\left(x^2+y^2\right)=a^2y^2\Rightarrow y^2\left(a^2-x^2\right)=x^4$	**M1**
	$y=\frac{x^2}{\sqrt{\left(a^2-x^2\right)}}$ **AG**	**A1**
		3
5(d)	$\frac{1}{2}\left(a\cos\frac{1}{4}\pi\right)\left(a\sin\frac{1}{4}\pi\right)-\frac{1}{2}a^2\left(1-\frac{1}{4}\pi\right)=\frac{1}{4}a^2\left(\frac{1}{2}\pi-1\right)$ **(A1 FT** *their* **(b))**	**M1 A1 FT**
		11

5. Polar coordinates

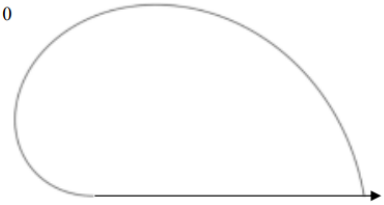
Q41.

7(a)	$r^5 = 4xy\left(x^2 - y^2\right)$	**B1**	Uses $r^2 = x^2 + y^2$.
	$r^5 = 4r^4 \cos\theta \sin\theta\left(\cos^2\theta - \sin^2\theta\right)$	**B1**	Uses $x = r\cos\theta$ and $y = r\sin\theta$.
	$r = 2\sin 2\theta \cos 2\theta$	**M1**	Applies at least one double angle formula.
	$r = \sin 4\theta$	**A1**	Applies both double angle formulae, AG.
		4	
7(b)		**B1**	Initial line drawn and one loop in the first quadrant.
		B1	Correct shape at extremities.
	$\theta = \frac{1}{8}\pi$	**B1**	States the equation of the line of symmetry.
		3	
7(c)	$\frac{1}{2}\int_0^{\frac{1}{4}\pi} \sin^2 4\theta \, d\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \, d\theta$ with correct limits.
	$= \frac{1}{4}\int_0^{\frac{1}{4}\pi} 1 - \cos 8\theta \, d\theta = \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\frac{1}{4}\pi}$	**M1 A1**	Applies double angle formula and integrates.
	$= \frac{1}{16}\pi$	**A1**	
		4	

7(d)	$x = 4\left(\sin\theta\cos^4\theta - \sin^3\theta\cos^2\theta\right)$	**B1**	Uses $x = r\cos\theta$.
	$-4\sin^2\theta\cos^3\theta + \cos^5\theta + 2\sin^4\theta\cos\theta - 3\sin^2\theta\cos^3\theta = 0$	**M1**	Differentiates and sets equal to 0.
	$\cos\theta\left(2\sin^4\theta - 7\sin^2\theta\cos^2\theta + \cos^4\theta\right) = 0$	**A1**	
	$\cos\theta = 0$ or $2\tan^4\theta - 7\tan^2\theta + 1 = 0$	**M1**	Forms quadratic in $\tan^2\theta$. Must see consideration of $\cos\theta = 0$.
	$\tan^2\theta = \frac{1}{4}\left(7 \pm \sqrt{41}\right) \Rightarrow \theta = \pm 0.369, \pm 1.071$	**B1**	Allow use of decimals.
	$\theta = 0.369 \Rightarrow x = 0.93$ (or $\theta = -0.369 \Rightarrow \lvert x \rvert = 0.93$)	**A1**	Substituting $\theta = \pm 0.369$ gives maximum value of $\lvert x \rvert$.
		6	

5. Polar coordinates

Q42.

5(a)	$\theta = 0$ 	**B1**	Correct shape, r decreasing.
		B1	Section $\frac{1}{2}\pi \le \theta \le \pi$ correct, becoming tangential at $\theta = \pi$.
	$(\ln(1+\pi), 0)$	**B1**	May be seen on their diagram. Allow (1.42, 0)
		3	
5(b)	$\frac{1}{2}\int_0^{\pi} \ln^2(1+\pi-\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_1^{1+\pi} \ln^2 u\,\mathrm{d}u$	**M1**	Uses correct formula with correct limits.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \int_1^{1+\pi} \ln u\,du$	**M1 A1**	Integrates by parts once.
	$= \left[\frac{1}{2}u\ln^2 u\right]_1^{1+\pi} - \left(\left[u\ln u\right]_1^{1+\pi} - \int_1^{1+\pi} 1\,\mathrm{d}u\right)$	**M1**	Integrates by parts again (or uses known result for integral of ln u)
	$= \left[\frac{1}{2}u\ln^2 u - u\ln u + u\right]_1^{1+\pi}$	**A1**	
	$\frac{1}{2}(1+\pi)\ln^2(1+\pi) - (1+\pi)\ln(1+\pi) + \pi =$ $\frac{1}{2}(1+\pi)\ln(1+\pi)(\ln(1+\pi)-2) + \pi$	**A1**	AG
		6	

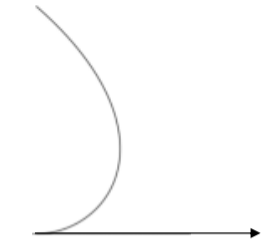
5(c)	$y = \ln(1+\pi-\theta)\sin\theta$	**B1**	Uses $y = r\sin\theta$
	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \ln(1+\pi-\theta)\cos\theta - \frac{\sin\theta}{1+\pi-\theta} = 0$	**M1 A1**	Sets derivative equal to zero.
	$\cos\theta \neq 0 \Rightarrow (1+\pi-\theta)\ln(1+\pi-\theta) - \tan\theta = 0$	**A1**	AG
	$(1+\pi-1.2)\ln(1+\pi-1.2) - \tan 1.2 = 0.602$ and $(1+\pi-1.3)\ln(1+\pi-1.3) - \tan 1.3 = -0.634$	**B1**	Shows sign change. Values correct to 2sf should be shown.
		5	

5. Polar coordinates

Q43.

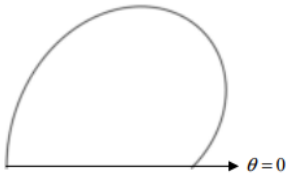
5(a)	$\theta = 0$ 	**B1 B1**	Initial line with correct position and shape of curve. SCB1 if correct shape with wrong starting position. Pole position must be clear.
	$a\cot\frac{1}{6}\pi = 2\sqrt{3} \Rightarrow a = 2$	**B1**	Substitutes $\theta = \frac{1}{6}\pi$. AG.
		3	
5(b)	$\frac{1}{2}\int_0^{\frac{1}{6}\pi} 4\cot^2\left(\frac{1}{3}\pi - \theta\right) \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$ with correct limits.
	$2\int_0^{\frac{1}{6}\pi} \operatorname{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1 \,\mathrm{d}\theta = 2\left[\cot\left(\frac{1}{3}\pi - \theta\right) - \theta\right]_0^{\frac{1}{6}\pi}$	**M1 A1**	Uses $\cot^2\left(\frac{1}{3}\pi - \theta\right) = \operatorname{cosec}^2\left(\frac{1}{3}\pi - \theta\right) - 1$ and integrates.
	$2\left(\sqrt{3} - \frac{1}{6}\pi - \frac{1}{3}\sqrt{3}\right) = \frac{4}{3}\sqrt{3} - \frac{1}{3}\pi$	**A1**	OE, must be exact.
		4	
5(c)	$r = 2\dfrac{\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$	**M1**	Uses the identities for $\cos(A-B)$ and $\sin(A-B)$. Or identity for $\tan(A-B)$.
	$\sqrt{x^2+y^2} = 2\dfrac{r\cos\theta + \sqrt{3}r\sin\theta}{\sqrt{3}r\cos\theta - r\sin\theta}$	**M1**	Applies $r = \sqrt{x^2+y^2}$, $x = r\cos\theta$ and $y = r\sin\theta$.
	$2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2+y^2}$	**A1**	AG.
		3	

Q44.

5(a)	 [O] [$\theta = 0$]	**B1**	Approximately correct curve passing through the pole, O, with correct domain.
		B1	r strictly increasing over the domain 0 to $\frac{\pi}{2}$.
		B1	Correct form at O. Approx tangential to initial line at O.
		3	
5(b)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}\left(\frac{1}{\pi-\theta} - \frac{1}{\pi}\right)^2 \mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2 \,\mathrm{d}\theta$ with correct limits.
	$\frac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{1}{(\pi-\theta)^2} - \frac{2}{\pi(\pi-\theta)} + \frac{1}{\pi^2} \mathrm{d}\theta$	**M1**	Expands.
	$\frac{1}{2}\left[\frac{1}{\pi-\theta} + \frac{2}{\pi}\ln(\pi-\theta) + \frac{\theta}{\pi^2}\right]_0^{\frac{1}{2}\pi}$	**M1 A1**	Integrates all terms to obtain correct form.
	$\frac{1}{2}\left(\frac{2}{\pi} + \frac{2}{\pi}\ln\frac{\pi}{2} + \frac{1}{2\pi} - \left(\frac{1}{\pi} + \frac{2}{\pi}\ln\pi\right)\right) = \frac{1}{2}\left(\frac{3}{2\pi} + \frac{2}{\pi}\ln\frac{1}{2}\right)$	**M1**	Substitute limits into an expression of the correct form and simplify the log terms.
	$= \dfrac{3 - 4\ln 2}{4\pi}$	**A1**	AG
		6	

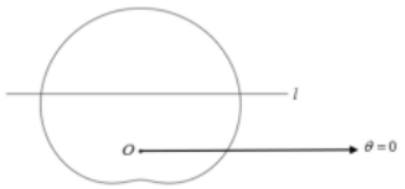
5. Polar coordinates

Q45.

6(a)	$2(c(c)-(1+s)s)=0$	**M1**	Finds $\frac{dr}{d\theta}$ and sets equal to 0.
	$1-s-2s^2=0$	**A1**	
	$\sin\theta=-1,\ \sin\theta=\frac{1}{2}$ leading to $\theta=\frac{1}{6}\pi$	**M1 A1**	Solves quadratic in $\sin\theta$.
	$\left(\frac{3}{2}\sqrt{3},\frac{1}{6}\pi\right)$	**A1**	Allow $\left(2.60,\frac{1}{6}\pi\right)$ 3sf
		5	
6(b)		**B1**	Correct shape. Tangential to $\theta=\frac{\pi}{2}$ at pole, r strictly increasing to $\frac{\pi}{6}$ then decreasing.
		B1	Correct position. All in first quadrant.
		2	

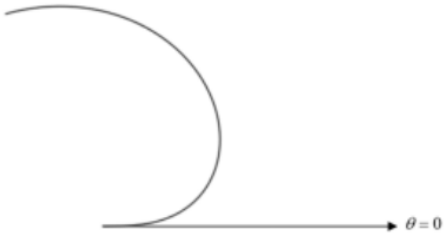
6(c)	$\frac{1}{2}\int_0^{\frac{1}{2}\pi}4c^2(1+s)^2\,d\theta=2\int_0^{\frac{1}{2}\pi}c^2+2sc^2+s^2c^2\,d\theta$	**M1 A1**	Uses $\frac{1}{2}\int r^2\,d\theta$ with correct limits.
	$2\int_0^{\frac{1}{2}\pi}\frac{1}{2}(\cos 2\theta+1)+2\sin\theta\cos^2\theta+\frac{1}{4}\sin^2 2\theta\,d\theta$	**M1**	Uses double angle formulae on first and last terms.
	$\int_0^{\frac{1}{2}\pi}\cos 2\theta+1+4\sin\theta\cos^2\theta+\frac{1}{4}-\frac{1}{4}\cos 4\theta\,d\theta$	**A1**	Obtains integrable form.
	$\left[\frac{5}{4}\theta+\frac{1}{2}\sin 2\theta-\frac{4}{3}\cos^3\theta-\frac{1}{16}\sin 4\theta\right]_0^{\frac{1}{2}\pi}$	**M1**	Integrates.
	$\frac{5}{8}\pi+\frac{4}{3}$	**A1**	
		6	

Q46.

5(a)		**B1**	Correct symmetrical shape, closed loop.
		1	
5(b)	Line l parallel to initial line and correct side of pole.	**B1**	
	$2=3\sin\theta+2\sin^2\theta$	**M1**	Forms quadratic in $\sin\theta$. Or in r $r=3+\frac{4}{r}$
	$\sin\theta=\frac{1}{2}$	**M1**	Solves for $\sin\theta$.
	$\left(4,\frac{1}{6}\pi\right)$ $\left(4,\frac{5}{6}\pi\right)$	**A1** **A1**	**SC1** For finding both angles correctly
		5	

5. Polar coordinates

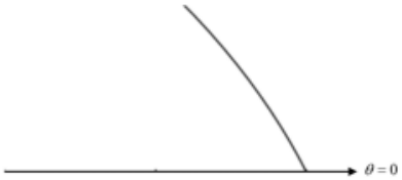
Q47.

6(a)		**B1**	Correct domain, r strictly increasing.
		B1	Correct gradient when $\theta = 0$ and $\theta = 2$.
	Maximum distance of C from the pole is $\sqrt{\frac{1}{4}\pi}$.	**B1**	Must be exact. $\frac{1}{2}\sqrt{\pi}$
		3	
6(b)	$\frac{1}{2}\int_0^2 \tan^{-1}\left(\frac{1}{2}\theta\right)\mathrm{d}\theta$	**M1**	Uses $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$\left[\frac{1}{2}\theta\tan^{-1}\left(\frac{1}{2}\theta\right)\right]_0^2 - \int_0^2 \frac{\theta}{\theta^2+4}\,\mathrm{d}\theta$	**M1 A1**	Applies integration by parts.
	$\frac{1}{2}\left[\theta\tan^{-1}\left(\frac{1}{2}\theta\right) - \ln(\theta^2+4)\right]_0^2$	**A1**	
	$\frac{1}{4}\pi - \frac{1}{2}\ln 8 + \frac{1}{2}\ln 4 = \frac{1}{4}\pi - \frac{1}{2}\ln 2$	**A1**	Must be exact.
		5	

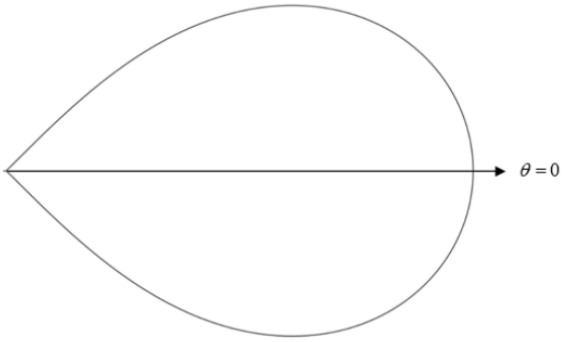
6(c)	$x = \left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{\frac{1}{2}}\cos\theta$	**B1**	Uses $x = r\cos\theta$
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{\frac{1}{2}}\sin\theta + \cos\theta\left(\tan^{-1}\left(\frac{1}{2}\theta\right)\right)^{-\frac{1}{2}}\left(\theta^2+4\right)^{-1} = 0$	**M1 A1**	Sets derivative equal to zero.
	$\left(\theta^2+4\right)\tan^{-1}\left(\frac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$	**A1**	AG
	$\left(0.6^2+4\right)\tan^{-1}(0.3)\sin 0.6 - \cos 0.6 = -0.108$ and $\left(0.7^2+4\right)\tan^{-1}(0.35)\sin 0.7 - \cos 0.7 = 0.209$	**B1**	Shows sign change.
		5	

5. Polar coordinates

Q48.

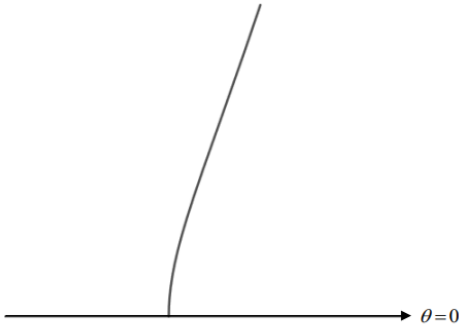
6(a)	$x=r\cos\theta \quad y=r\sin\theta \quad x^2+y^2=r^2$	**B1**	SOI
	$r^2\left(1+\sin\theta\cos\theta\right)=r^2\left(1+\frac{1}{2}\sin 2\theta\right)=a$	**M1**	Eliminates both x and y and uses the double angle formula.
	$r^2=\dfrac{2a}{2+\sin 2\theta}$	**A1**	AG
		3	
6(b)		**B1**	Polar graph with curve in correct domain. Graph needs gradient $\leqslant 0$ and $r>0$ for all θ in the domain.
		B1	r strictly decreasing such that r at $\theta=\frac{\pi}{4}$ is greater than half r at $\theta=0$. Concave graph. Gradient at $\theta=0$ and $\frac{\pi}{4}$ must not be vertical or horizontal, respectively.
		2	

Q49.

6(a)	$x^2+y^2=r^2 \quad x=r\cos\theta \quad y=r\sin\theta$	**B1**	Used.
	$r^4=36r^2\left(\cos^2\theta-\sin^2\theta\right)=36r^2\cos 2\theta$	**M1**	Substitutes and applies $\cos 2\theta=\cos^2\theta-\sin^2\theta$.
	$r^2=36\cos 2\theta$	**A1**	AG.
		3	
6(b)		**B1**	Closed curve. Correct position and symmetrical about initial line.
		B1	Single correct loop.
	6	**B1**	States maximum distance or labels sketch.
		3	

6(c)	$18\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta\, d\theta = 9[\sin 2\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$	**M1**	Forms $\frac{1}{2}\int r^2\, d\theta$
	18	**A1**	
		2	
6(d)	$y = 6\cos^{\frac{1}{2}} 2\theta \sin\theta$	**B1**	
	$\cos^{\frac{1}{2}} 2\theta\cos\theta - \cos^{-\frac{1}{2}} 2\theta \sin 2\theta \sin\theta = 0$	**M1 A1**	Sets $\frac{dy}{d\theta} = 0$.
	$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$ leading to $1 = \frac{2\tan^2\theta}{1-\tan^2\theta}$	**M1**	Applies suitable trigonometric identity.
	$\theta = \pm\frac{1}{6}\pi$	**A1**	
	$\frac{3}{2}\sqrt{2}$	**A1**	
		6	

Q50.

5(a)		**B1**	Correct position and domain and r strictly increasing.
		B1	Decreasing positive gradient.
	$(a, 0)$ $(2a, \frac{1}{4}\pi)$	**B1**	Can be labelled on their sketch.
		3	
5(b)	$y = (2a)\sin\left(\frac{1}{4}\pi\right)$	**M1**	Uses $y = r\sin\theta$
	$a\sqrt{2}$	**A1**	
		2	

5. Polar coordinates

5(c)	$\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^4\theta\,\mathrm{d}\theta$	**M1**	Forms $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ with correct limits.
	$=\frac{1}{2}a^2\int_0^{\frac{1}{4}\pi}\sec^2\theta\left(\tan^2\theta+1\right)\mathrm{d}\theta$	**M1**	Applies $\sec^2\theta=\tan^2\theta+1$ to obtain an integrable form.
	$=\frac{1}{2}a^2\left[\frac{1}{3}\tan^3\theta+\tan\theta\right]_0^{\frac{1}{4}\pi}$	**A1**	
	$=\frac{2}{3}a^2$	**A1**	
		4	
5(d)	$x^2+y^2=r^2 \quad x=r\cos\theta$	**M1**	Substitutes to eliminate either r or θ.
	$r^2\cos^2\theta=ar$ leading to $x^2=a\sqrt{x^2+y^2}$ leading to $x^4=a^2\left(x^2+y^2\right)$	**M1**	Completes substitution to eliminate both r and θ and makes y or y^2 the subject.
	$y=\sqrt{a^{-2}x^4-x^2}=x\sqrt{a^{-2}x^2-1}$	**A1**	AEF
		3	

6. Vectors

Q1.

The position vectors of the points A, B, C, D are

$$7\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \qquad 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}, \qquad 2\mathbf{i} + 7\mathbf{j} + \lambda\mathbf{k}$$

respectively. It is given that the shortest distance between the line AB and the line CD is 3.

(i) Show that $\lambda^2 - 5\lambda + 4 = 0$. [7]

(ii) Find the acute angle between the planes through A, B, D corresponding to the values of λ satisfying the equation in part **(i)**. [7]

9231/01/M/J/08 Q12

Q2.

The plane Π_1 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(2\mathbf{j} - \mathbf{k}) + \phi(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a vector normal to Π_1 and hence show that the equation of Π_1 can be written as $2x + 3y + 6z = 14$. [4]

The line l has equation

$$\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}).$$

The point on l where $t = \lambda$ is denoted by P. Find the set of values of λ for which the perpendicular distance of P from Π_1 is not greater than 4. [4]

The plane Π_2 contains l and the point with position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the acute angle between Π_1 and Π_2. [4]

9231/01/O/N/08 Q11

Q3.

The line l_1 is parallel to the vector $4\mathbf{j} - \mathbf{k}$ and passes through the point A whose position vector is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. The variable line l_2 is parallel to the vector $\mathbf{i} - (2\sin t)\mathbf{j}$, where $0 \leqslant t < 2\pi$, and passes through the point B whose position vector is $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The points P and Q are on l_1 and l_2, respectively, and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ in terms of t. [5]

(ii) Hence find the values of t for which l_1 and l_2 intersect. [2]

(iii) For the case $t = \frac{1}{4}\pi$, find the perpendicular distance from A to the plane BPQ, giving your answer correct to 3 decimal places. [5]

9231/01/M/J/09 Q11

6. Vectors

Q4.

Relative to an origin O, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{j}+\mathbf{k}, \quad \mathbf{i}+\mathbf{j}+\theta\mathbf{k},$$

respectively. The shortest distance between the lines AB and OC is $\frac{1}{\sqrt{2}}$. Find the value of θ. [6]

9231/01/O/N/09 Q2

Q5.

The lines l_1 and l_2 have vector equations

$$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

respectively.

(i) Show that l_1 and l_2 intersect. [3]

(ii) Find the perpendicular distance from the point P whose position vector is $3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ to the plane containing l_1 and l_2. [3]

(iii) Find the perpendicular distance from P to l_1. [4]

9231/11/M/J/10 Q7

Q6.

The line l_1 passes through the point A whose position vector is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and is parallel to the vector $\mathbf{i} + \mathbf{j}$. The line l_2 passes through the point B whose position vector is $-\mathbf{i} - \mathbf{k}$ and is parallel to the vector $\mathbf{j} + 2\mathbf{k}$. The point P is on l_1 and the point Q is on l_2 and PQ is perpendicular to both l_1 and l_2.

(i) Find the length of PQ. [4]

(ii) Find the position vector of Q. [5]

(iii) Show that the perpendicular distance from Q to the plane containing AB and the line l_1 is $\sqrt{3}$. [4]

9231/13/M/J/10 Q12

6. Vectors

Q7.

The plane Π_1 has equation $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) + \mu(-\mathbf{i} + \mathbf{k})$. Obtain a cartesian equation of Π_1 in the form $px + qy + rz = d$. [4]

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = 12$. Find a vector equation of the line of intersection of Π_1 and Π_2. [3]

The line l passes through the point A with position vector $a\mathbf{i} + (2a + 1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3c\mathbf{i} - 3\mathbf{j} + c\mathbf{k}$, where a and c are positive constants. Given that the perpendicular distance from A to Π_1 is $\frac{15}{\sqrt{6}}$ and that the acute angle between l and Π_1 is $\sin^{-1}\left(\frac{2}{\sqrt{6}}\right)$, find the values of a and c. [7]

9231/01/O/N/10 Q12

Q8.

The line l_1 passes through the point with position vector $8\mathbf{i} + 8\mathbf{j} - 7\mathbf{k}$ and is parallel to the vector $4\mathbf{i} + 3\mathbf{j}$. The line l_2 passes through the point with position vector $7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and is parallel to the vector $4\mathbf{i} - \mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. In either order,

(i) show that $PQ = 13$,

(ii) find the position vectors of P and Q.

[9]

9231/11/M/J/11 Q6

Q9.

The lines l_1 and l_2 have equations

$$l_1: \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad l_2: \mathbf{r} = 6\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \mu(4\mathbf{i} + 6\mathbf{j} + \mathbf{k}).$$

Find a cartesian equation of the plane Π containing l_1 and l_2. [4]

Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{i} + 10\mathbf{j} + 3\mathbf{k}$ to Π. [4]

The line l_3 has equation $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + \nu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$. Find the shortest distance between l_1 and l_3. [5]

9231/13/M/J/11 Q10

6. Vectors

Q10.

The position vectors of points A, B, C, relative to the origin O, are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \qquad \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \qquad \mathbf{c} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{a} \times \mathbf{b}$ and deduce the area of the triangle OAB. [3]

Hence find the volume of the tetrahedron $OABC$, given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height. [2]

9231/11/O/N/11 Q2

Q11.

Find a cartesian equation of the plane Π containing the lines

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + s(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + 10\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}).$$ [4]

The line l passes through the point P with position vector $6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and is parallel to the vector $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Find

(i) the position vector of the point where l meets Π, [3]

(ii) the perpendicular distance from P to Π, [3]

(iii) the acute angle between l and Π. [3]

9231/13/O/N/11 Q9

Q12.

The position vectors of the points A, B, C, D are

$$2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}, \quad -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}, \quad \mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \mathbf{i} + 5\mathbf{j} + m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3. Show that the only possible value of m is 2. [7]

Find the shortest distance of D from the line through A and C. [3]

Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

9231/11/M/J/12 Q11

6. Vectors

Q13.

The plane Π_1 has parametric equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a cartesian equation of Π_1. [4]

The plane Π_2 has cartesian equation $3x - 2y - 3z = 4$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

9231/13/M/J/12 Q9

Q14.

The plane Π has equation

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + \mu(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

The line l, which does not lie in Π, has equation

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} + t(8\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}).$$

Show that l is parallel to Π. [4]

Find the position vector of the point at which the line with equation $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + s(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ meets Π. [4]

Find the perpendicular distance from the point with position vector $9\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}$ to l. [4]

9231/11/O/N/12 Q9

Q15.

The points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

Deduce, in either order, the exact value of

(i) the area of the triangle ABC,

(ii) the perpendicular distance from C to AB.

[3]

9231/13/O/N/12 Q4

6. Vectors

Q16.

The points A, B, C and D have coordinates as follows:

$$A\,(2,\ 1,\ -2), \quad B\,(4,\ 1,\ -1), \quad C\,(3,\ -2,\ -1) \quad \text{and} \quad D\,(3,\ 6,\ 2).$$

The plane Π_1 passes through the points A, B and C. Find a cartesian equation of Π_1. [4]

Find the area of triangle ABC and hence, or otherwise, find the volume of the tetrahedron $ABCD$. [6]

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.]

The plane Π_2 passes through the points A, B and D. Find the acute angle between Π_1 and Π_2. [4]

9231/11/M/J/13 Q11

Q17.

The line l_1 passes through the point A whose position vector is $4\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ and is parallel to the vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The line l_2 passes through the point B whose position vector is $\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$ and is parallel to the vector $\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$. The points P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

Find the shortest distance between the line through A and B and the line through P and Q, giving your answer correct to 3 significant figures. [6]

9231/13/M/J/13 Q11

Q18.

The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $2x - y + z = 10$. Find the acute angle between Π_1 and Π_2. [2]

Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

9231/11/O/N/13 Q8

Q19.

The points A, B, C have position vectors

$$4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}, \qquad 5\mathbf{i} + 7\mathbf{j} + 8\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 4\mathbf{k},$$

respectively, relative to the origin O. Find a cartesian equation of the plane ABC. [4]

The point D has position vector $6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. Find the coordinates of E, the point of intersection of the line OD with the plane ABC. [4]

Find the acute angle between the line ED and the plane ABC. [3]

9231/13/O/N/13 Q8

6. Vectors

Q20.

The line l_1 passes through the points $A\,(2,\ 3,\ -5)$ and $B\,(8,\ 7,\ -13)$. The line l_2 passes through the points $C\,(-2,\ 1,\ 8)$ and $D\,(3,\ -1,\ 4)$. Find the shortest distance between the lines l_1 and l_2. [5]

The plane Π_1 passes through the points A, B and D. The plane Π_2 passes though the points A, C and D. Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [6]

9231/11/M/J/14 Q11

Q21.

With respect to an origin O, the point A has position vector $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and the plane Π_1 has equation

$$\mathbf{r} = (4 + \lambda + 3\mu)\mathbf{i} + (-2 + 7\lambda + \mu)\mathbf{j} + (2 + \lambda - \mu)\mathbf{k},$$

where λ and μ are real. The point L is such that $\overrightarrow{OL} = 3\overrightarrow{OA}$ and Π_2 is the plane through L which is parallel to Π_1. The point M is such that $\overrightarrow{AM} = 3\overrightarrow{ML}$.

(i) Show that A is in Π_1. [1]

(ii) Find a vector perpendicular to Π_2. [2]

(iii) Find the position vector of the point N in Π_2 such that ON is perpendicular to Π_2. [5]

(iv) Show that the position vector of M is $10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ and find the perpendicular distance of M from the line through O and N, giving your answer correct to 3 significant figures. [6]

9231/13/M/J/14 Q11

Q22.

The line l_1 is parallel to the vector $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and passes through the point A, whose position vector is $3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. The line l_2 is parallel to the vector $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and passes through the point B, whose position vector is $-3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find

(i) the length PQ, [5]

(ii) the cartesian equation of the plane Π containing PQ and l_2, [4]

(iii) the perpendicular distance of A from Π. [3]

9231/11/O/N/14 Q10

6. Vectors

Q23.

The lines l_1 and l_2 have equations $\mathbf{r} = 8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j})$ and $\mathbf{r} = 5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k} + \mu(2\mathbf{j} - 3\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vector of the point P and the position vector of the point Q. [8]

The points with position vectors $8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $5\mathbf{i} + 3\mathbf{j} - 14\mathbf{k}$ are denoted by A and B respectively. Find

(i) $\overrightarrow{AP} \times \overrightarrow{AQ}$ and hence the area of the triangle APQ,

(ii) the volume of the tetrahedron $APQB$. (You are given that the volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.)

[6]

9231/11/M/J/15 Q11

Q24.

A line, passing through the point $A\,(3,\ 0,\ 2)$, has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point P. Find the coordinates of P. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point Q lies in Π and is the foot of the perpendicular from A to Π. Use the vector $\mathbf{w}$ to determine an equation of the line PQ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

9231/13/M/J/15 Q8

Q25.

The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\dfrac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

9231/11/O/N/15 Q11

6. Vectors

Q26.

Find a cartesian equation of the plane Π_1 passing through the points with coordinates (2, −1, 3), (4, 2, −5) and (−1, 3, −2). [4]

The plane Π_2 has cartesian equation $3x - y + 2z = 5$. Find the acute angle between Π_1 and Π_2. [3]

Find a vector equation of the line of intersection of the planes Π_1 and Π_2. [4]

9231/11/M/J/16 Q8

Q27.

The position vectors of the points A, B, C, D are

$$\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}, \quad \mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{d} = \mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$$

respectively. It is given that the shortest distance between the lines AB and CD is 3.

(i) Show that $\lambda^2 + \lambda - 20 = 0$. [7]

(ii) The planes p_1 and p_2 are the planes through A, B and D corresponding to the two values of λ satisfying the equation in part **(i)**. Find the acute angle between p_1 and p_2. [7]

9231/13/M/J/16 Q11

Q28.

The lines l_1 and l_2 have equations

$$\mathbf{r} = 6\mathbf{i} - 3\mathbf{j} + s(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$$

respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Show that the position vector of P is $3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and find the position vector of Q. [7]

Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane Π which passes through P and is perpendicular to l_1. [3]

The plane Π meets the plane $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$ in the line l_3. Find a vector equation of l_3. [4]

9231/11/O/N/16 Q11

6. Vectors

Q29.

The position vectors of the points A, B, C, D are

$$\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad 3\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad 5\mathbf{i} - 5\mathbf{j} + \alpha\mathbf{k},$$

respectively, where α is a positive integer. It is given that the shortest distance between the line AB and the line CD is equal to $2\sqrt{2}$.

(i) Show that the possible values of α are 3 and 5. [7]

(ii) Using $\alpha = 3$, find the shortest distance of the point D from the line AC, giving your answer correct to 3 significant figures. [3]

(iii) Using $\alpha = 3$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [4]

9231/11/M/J/17 Q12

Q30.

The plane Π_1 passes through the points (1, 2, 1) and (5, −2, 9) and is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(i) Find the cartesian equation of Π_1. [4]

The plane Π_2 contains the lines

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}).$$

(ii) Find the cartesian equation of Π_2. [4]

(iii) Find the acute angle between Π_1 and Π_2. [3]

9231/13/M/J/17 Q9

Q31.

The points A, B and C have position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ respectively.

(i) Find the area of the triangle ABC. [4]

(ii) Find the perpendicular distance of the point A from the line BC. [3]

(iii) Find the cartesian equation of the plane through A, B and C. [2]

9231/11/O/N/17 Q6

6. Vectors

Q32.

The line l_1 is parallel to the vector $a\mathbf{i} - \mathbf{j} + \mathbf{k}$, where a is a constant, and passes through the point whose position vector is $9\mathbf{j} + 2\mathbf{k}$. The line l_2 is parallel to the vector $-a\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and passes through the point whose position vector is $-6\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$.

(i) It is given that l_1 and l_2 intersect.

(a) Show that $a = -\frac{6}{13}$. [3]

(b) Find a cartesian equation of the plane containing l_1 and l_2. [4]

(ii) Given instead that the perpendicular distance between l_1 and l_2 is $3\sqrt{(30)}$, find the value of a. [5]

9231/11/M/J/18 Q10

Q33.

The lines l_1 and l_2 have vector equations

$$\mathbf{r} = a\mathbf{i} + 9\mathbf{j} + 13\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = -3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

respectively. It is given that l_1 and l_2 intersect.

(i) Find the value of the constant a. [3]

The point P has position vector $3\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

(ii) Find the perpendicular distance from P to the plane containing l_1 and l_2. [4]

(iii) Find the perpendicular distance from P to l_2. [4]

9231/13/M/J/18 Q7

Q34.

The plane Π_1 has equation

$$\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Find a cartesian equation of Π_1. [3]

The plane Π_2 has equation $3x + y - z = 3$.

(ii) Find the acute angle between Π_1 and Π_2, giving your answer in degrees. [2]

(iii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. [5]

9231/11/O/N/18 Q8

6. Vectors

Q35.

The position vectors of the points A, B, C, D are

$$\mathbf{i}+\mathbf{j}+3\mathbf{k}, \qquad 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}, \qquad -\mathbf{i}+3\mathbf{k}, \qquad m\mathbf{j}+4\mathbf{k},$$

respectively, where m is a constant.

(i) Show that the lines AB and CD are parallel when $m = \frac{3}{2}$. [1]

(ii) Given that $m \neq \frac{3}{2}$, find the shortest distance between the lines AB and CD. [5]

(iii) When $m = 2$, find the acute angle between the planes ABC and ABD, giving your answer in degrees. [6]

9231/12/O/N/18 Q10

Q36.

The lines l_1 and l_2 have equations $\mathbf{r} = 6\mathbf{i}+2\mathbf{j}+7\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j})$ and $\mathbf{r} = 4\mathbf{i}+4\mathbf{j}+\mu(-6\mathbf{j}+\mathbf{k})$ respectively. The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. Find the position vectors of P and Q. [8]

9231/11/M/J/19 Q3

Q37.

The line l_1 passes through the points $A\,(-3, 1, 4)$ and $B\,(-1, 5, 9)$. The line l_2 passes through the points $C\,(-2, 6, 5)$ and $D\,(-1, 7, 5)$.

(i) Find the shortest distance between the lines l_1 and l_2. [5]

(ii) Find the acute angle between the line l_2 and the plane containing A, B and D. [5]

9231/13/M/J/19 Q7

Q38.

With O as the origin, the points A, B, C have position vectors

$$\mathbf{i}-\mathbf{j}, \qquad 2\mathbf{i}+\mathbf{j}+7\mathbf{k}, \qquad \mathbf{i}-\mathbf{j}+\mathbf{k}$$

respectively.

(i) Find the shortest distance between the lines OC and AB. [5]

(ii) Find the cartesian equation of the plane containing the line OC and the common perpendicular of the lines OC and AB. [4]

9231/11/O/N/19 Q6

6. Vectors

Q39.

The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i} + \mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

9231/11/M/J/20 Q5

Q40.

The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

9231/13/M/J/20 Q7

Q41.

The points A, B, C have position vectors

$$-\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad -2\mathbf{i} - \mathbf{j}, \quad 2\mathbf{i} + 2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

9231/11/O/N/20 Q4

6. Vectors

Q42.

The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the acute angle between the planes OBC and ABC. [4]

The point D has position vector $t\mathbf{i}-\mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

9231/12/O/N/20 Q7

Q43.

The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q.

(i) Find a vector perpendicular to the plane ABC and show that the length of ON is $\frac{4}{\sqrt{(21)}}$. [4]

(ii) Find the position vector of the point Q. [5]

(iii) Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

9231/01/SP/17 Q11

Q44.

The position vectors of the points A, B, C, D are

$$2\mathbf{i}+4\mathbf{j}-3\mathbf{k}, \quad -2\mathbf{i}+5\mathbf{j}-4\mathbf{k}, \quad \mathbf{i}+4\mathbf{j}+\mathbf{k}, \quad \mathbf{i}+5\mathbf{j}+m\mathbf{k},$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3.

(a) Show that the only possible value of m is 2. [7]

(b) Find the shortest distance of D from the line through A and C. [3]

(c) Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [4]

9231/01/SP/20 Q6 iii

6. Vectors

Q45.

Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

The plane Π_2 has Cartesian equation $5x-6y+7z=0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

9231/11/M/J/21 Q6

Q46.

The lines l_1 and l_2 have equations $\mathbf{r}=-\mathbf{i}-2\mathbf{j}+\mathbf{k}+s(2\mathbf{i}-3\mathbf{j})$ and $\mathbf{r}=3\mathbf{i}-2\mathbf{k}+t(3\mathbf{i}-\mathbf{j}+3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [2]

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

(d) The point Q is such that $\overrightarrow{OQ}=-5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

9231/13/M/J/21 Q6

6. Vectors

Q47.

The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

9231/11/O/N/21 Q5

Q48.

The points A, B, C have position vectors

$$2\mathbf{i} + 2\mathbf{j}, \quad -\mathbf{j} + \mathbf{k} \quad \text{and} \quad 2\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [3]

The plane Π has equation $x - 3y - 2z = 1$.

(b) Find the perpendicular distance of Π from the origin. [1]

(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

9231/12/O/N/21 Q7

Q49.

The points A, B, C have position vectors

$$4\mathbf{i} - 4\mathbf{j} + \mathbf{k}, \quad -4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad 4\mathbf{i} - \mathbf{j} - 2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) The point D has position vector $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

9231/11/M/J/22 Q2

6. Vectors

Q50.

The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda=1$.

Let Π_2 be the plane ABD when $\lambda=4$.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+s\mathbf{b}+t\mathbf{c}$. [2]

(ii) Find an equation of Π_2, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

9231/13/M/J/22 Q7

Q51.

The plane Π contains the lines $\mathbf{r}=3\mathbf{i}-2\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ and $\mathbf{r}=4\mathbf{i}+4\mathbf{j}+2\mathbf{k}+\mu(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

The line l passes through the point P with position vector $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and is parallel to the vector $\mathbf{j}+\mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

9231/11/O/N/22 Q4

6. Vectors

Q52.

The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

9231/12/O/N/22 Q6

Solutions

Q1.

(i) $\overrightarrow{AB} \times \overrightarrow{CD} = (\lambda - 2)\mathbf{i} + (4\lambda - 12)\mathbf{j} - 4\mathbf{k}$ M1A1

$(-5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}).[(\lambda - 2)\mathbf{i} + (4\lambda - 12)\mathbf{j} - 4\mathbf{k}] = 3\lambda - 30$ M1A1

$|3\lambda - 30|/\sqrt{(\lambda - 2)^2 + (4\lambda - 12)^2 + 16} = 3$ M1A1

$\lambda^2 - 5\lambda + 4 = 0$ (AG) A1

(ii) $\lambda - 1, 4$ B1

$\lambda = 1 : \mathbf{n}_1 = 5\mathbf{i} + 13\mathbf{j} - 7\mathbf{k}$ M1A1 cao

$\lambda = 4 : \mathbf{n}_2 = 8\mathbf{i} + 25\mathbf{j} - 7\mathbf{k}$ M1A1 cao

Acute angle between planes $= \cos^{-1}|\mathbf{n}_1.\mathbf{n}_2|/|\mathbf{n}_1||\mathbf{n}_2| = 12.1°$ M1A1 cao

[For lines 2 and 3: $\dfrac{10 - 5\lambda - 24 + 8\lambda - 16}{\sqrt{(2 - \lambda)^2 + (12 - 4\lambda)^2 + 4^2}} = 3$ M1A1

$(\lambda - 10)^2 = 164 - 100\lambda + 17\lambda^2$ M1A1]

Q2.

$(2\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -2\mathbf{i} - 3\mathbf{j} - 6\mathbf{j}$ (oew) M1A1

$\Pi_1 : 2x + 3y + 6z = 14$ (AG) M1A1

Perpendicular distance, p, of P from l in terms of 1 parameter, e.g.,

$p = (1/7)|2(3 + 4\lambda) + 3(8 + 6\lambda) + 6(2 + 5\lambda) - 14|$ M1

$= |4 + 8\lambda|$ A1

$p \leqslant 4 \Rightarrow -1 \leqslant \lambda \leqslant 0$ M1A1

$(3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 6\mathbf{j} + \mathbf{k}$

$(2\mathbf{i} + 6\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}) = 24\mathbf{i} - 6\mathbf{j} - 12\mathbf{k}$ M1A1

$\cos\alpha = |(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})/7\sqrt{21}| = 1/\sqrt{21}$ M1

$\alpha = 77.4°$ A1

Q3.

11 (i) $(\mathbf{i} - (2\sin t)\mathbf{j}) \times (4\mathbf{j} - \mathbf{k}) = (2\sin t)\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ M1A1

$PQ = (\mathbf{i} - \mathbf{j}).[(2\sin t)\mathbf{i} + \mathbf{j} + 4\mathbf{k}]/\sqrt{4\sin^2 t + 17}$ dM1A1

$PQ = |1 - 2\sin t|/\sqrt{4\sin^2 t + 17}$ A1

Condone disappearing modulus sign –

Deduct 1 mark if no modulus sign at all

(ii) $PQ = 0 \Rightarrow \sin t = 1/2 \Rightarrow t = \pi/6, 5\pi/6$ or 0.524, 2.62 M1A1 (both)

(iii) Obtains some vector $\perp$ plane BPQ, e.g.,

$(\sqrt{2}\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \sqrt{2}\mathbf{j}) = 4\sqrt{2}\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ M1A1

$p(A : BPQ) = (4\sqrt{2}\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}).(\mathbf{i} - \mathbf{j})/\sqrt{57}$ M1A1

$= 4(\sqrt{2} - 1)/\sqrt{57} = 0.219$ A1

or

For **(i)**

$$\ell_1 = r = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$$

$$\ell_2 = r = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2\sin t \\ 0 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} \mu - 1 \\ 1 - 4\lambda - 2\mu\sin t \\ \lambda \end{pmatrix}$$

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = 0 \qquad \Rightarrow 17\lambda + 8\mu\sin t = 4$$

$$PQ \cdot \begin{pmatrix} 1 \\ -2\sin t \\ 0 \end{pmatrix} = 0 \qquad \Rightarrow 8\sin t\lambda + (1 + 4\sin^2 t)\mu = 1 + 2\sin t$$ M1

Whence $\lambda = \dfrac{4(1 - 2\sin t)}{(17 + 4\sin^2 t)}$ and $\mu = \dfrac{17 + 2\sin t}{(17 + 4\sin^2 t)}$ A1 (both)

For $\overrightarrow{PQ}$ $x=\dfrac{2\sin t(1-2\sin t)}{(17+4\sin^2 t)}$ $y=\dfrac{1-2\sin t}{(17+4\sin^2 t)}$ $z=\dfrac{4(1-2\sin t)}{17+4\sin^2 t}$ dM1A1

(λ and μ of the form $\dfrac{a+b\sin t}{17+4\sin^2 t}$)

$$\left|\overrightarrow{PQ}\right|=\sqrt{x^2+y^2+z^2}=\left|\frac{(1-2\sin t)}{(17+4\sin^2 t)}\sqrt{4\sin^2 t+1^2+4^2}\right|$$

$$=\frac{|1-2\sin t|}{\sqrt{17+4\sin^2 t}}$$ A1

For **(iii)** Eliminates λ and μ from parametric equation of plane BPQ

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-\sqrt{2}\\0\end{pmatrix}+\mu\begin{pmatrix}\sqrt{2}\\1\\4\end{pmatrix}$$ M1A1

to obtain cartesian equation of plane BPQ

$4\sqrt{2}x+4y-3z=4(\sqrt{2}-1)$ * A1

Uses distance of point from a line formula with * and (2, 1, 4) M1

$$d=\frac{8\sqrt{2}+4-12-4\sqrt{2}+4}{\sqrt{32+16+9}}=\frac{4(\sqrt{2}-1)}{\sqrt{57}}$$

$=0.219$ cao A1

Q4.

2 Evaluates some relevant vector product, e.g.,
$(\mathbf{i}-\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\theta\mathbf{k})=(1-\theta)\mathbf{i}-(1+\theta)\mathbf{j}+2\mathbf{k}$ M1A1

$p=\left|[(1-\theta)\mathbf{i}-(1+\theta)\mathbf{j}+2\mathbf{k}].\mathbf{i}\right|/\sqrt{2\theta^2+6}$ M1

$=\left|(1-\theta)/\sqrt{2\theta^2+6}\right|$ A1

Puts $p=1/\sqrt{2}$ to obtain a horizontal equation such as
$2\theta^2+6=2(\theta-1)^2$ M1
$\Rightarrow\ldots\Rightarrow\theta=-1$ A1

Q5.

7 (i) Solves any 2 of the equations:
$4 + 2\lambda = 4 + \mu$, $-2 + \lambda = -5 - \mu$, $-4\lambda = 2 - \mu$
to obtain $\lambda = -1$, $\mu = -2$ M1A1
Checks consistency with the third equation A1
[3]

(ii) $P = |(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) / \sqrt{38}|$ M1A1
$= 7 / \sqrt{38} = 1.14$ A1
[3]

OR
$\mathbf{n} = -5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ M1
Plane is $5x + 2y + 3z = 16$ A1
$P = \dfrac{15 - 10 + 18 - 16}{\sqrt{5^2 + 2^2 + 3^2}} = \dfrac{7}{\sqrt{38}}$ A1

OR
Plane is $5x + 2y + 3z = 16$ (as above) M1A1
Sub. general pt on perpendicular $\begin{pmatrix} 3+5t \\ -5+2t \\ 6+3t \end{pmatrix} \Rightarrow t = -\dfrac{7}{38}$

$\Rightarrow P = \left|\begin{pmatrix} 5t \\ 2t \\ 3t \end{pmatrix}\right| = 1.14$ A1

(iii) $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$
OR $(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = -6\mathbf{i} - 8\mathbf{j} - 5\mathbf{k}$, etc. B1
$d = |6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}| / \sqrt{21} = \sqrt{125/21} = 2.44$ M1A1A1
[4]

OR
Let Q be the foot of the perpendicular from P to l, and A be the known point on l_1
$AQ = \left|(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).\dfrac{(2\mathbf{i} + \mathbf{j} - 4\mathbf{k})}{\sqrt{21}}\right| = \dfrac{8}{\sqrt{21}}$ M1A1
$AP^2 = 1^2 + (-2)^2 + 2^2 = 9$ B1
$PQ^2 = 9 - \dfrac{64}{21} = \dfrac{125}{21} \Rightarrow PQ = \dfrac{5\sqrt{5}}{21}$ A1

OR
$\overrightarrow{PQ} = \begin{pmatrix} 4+2t \\ -2+t \\ -4t \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1+2t \\ 3+t \\ -6-4t \end{pmatrix} \therefore \begin{pmatrix} 1+2t \\ 3+t \\ -6-4t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = 0$ M1
$\Rightarrow t = -\dfrac{29}{21}$ A1
$\overrightarrow{PQ} = \dfrac{1}{21}\begin{pmatrix} -37 \\ 34 \\ -10 \end{pmatrix} \Rightarrow \left|\overrightarrow{PQ}\right| = \dfrac{1}{21}\sqrt{37^2 + 34^2 + 10^2} = 2.44$ M1A1

6. Vectors

Q6.

12 EITHER

(i) $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}+2\mathbf{k}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ M1A1

$PQ = (4\mathbf{i}+\mathbf{j}+3\mathbf{k}).(2\mathbf{i}-2\mathbf{j}+\mathbf{k})/3 = 9/3 = 3$ M1A1
[4]

(ii) $(4+\lambda)/2 = (1+\lambda-\mu)/(-2) = (3-2\mu)/1$ (AEF)

OR $(4+\lambda)+(1+\lambda-\mu)=0$, $(1+\lambda-\mu+2(3-2\mu)=0$, both M1A1

$\Rightarrow \ldots \Rightarrow \mu = 1$ M1A1

Position vector of Q is $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ A1
[5]

Parts **(i)** and **(ii)** together:

$$\overrightarrow{PQ} = \begin{pmatrix} 4+\lambda \\ 1+\lambda-\mu \\ 3-2\mu \end{pmatrix}$$ B1

$$\overrightarrow{PQ}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow 5+2\lambda-\mu = 0$$ M1A1

$$\overrightarrow{PQ}.\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow 7+\lambda-5\mu = 0$$ A1

$\lambda = -2,\ \mu = 1$ M1A1

$$\overrightarrow{PQ}.\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \left|\overrightarrow{PQ}\right| = 3$$ M1A1

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$ A1

[9]

(iii) $(\mathbf{i}+\mathbf{j})\times(4\mathbf{i}+\mathbf{j}+3\mathbf{k}) = 3\mathbf{i}-3\mathbf{j}-3\mathbf{k}$ M1A1

$p_2 = \left|[(\mathbf{i}-\mathbf{j}-\mathbf{k}).(4\mathbf{i}+\mathbf{k})]/\sqrt{3}\right|$ M1

$= \ldots = \sqrt{3}$ A1

[4]

for final 2 marks

$\pi : x-y-z=0$

perpendicular distance $= \left|\left(\frac{-1-1-1}{\sqrt{1^2+1^2+1^2}}\right)\right| = \sqrt{3}$

or any other method.

M1 is for a complete strategy and A1 for $\sqrt{3}$.

Q7.

Normal to plane: $(2, 3, 4)\times(-1, 0, 1) = (3, -6, 3)$ M1A1

$\mathbf{r}.(1, -2, 1) = d$ and point $(2, 1, 4)$ M1 substitute point into plane eqn

$d = 4$ $x-2y+z=4$ A1 [4]

Alternative:

$x = 2+2\lambda-\mu$

$y = 1+2\lambda$ } $x+z = 6+6\lambda$ M1A1

$z = 4+4\lambda+\mu$

$\therefore x+z = 6+2(y-1)$ M1

$\therefore x-2y+z=4$ A1

$x-4y+5z=12$

$x-2y+z=4$ Solve by eliminating one variable M1

Use parameter and express all 3 variables in terms of it M1

e.g. $x = 3t-4$, $y = 2t-4$, $z = t$

$\mathbf{r} = (-4, -4, 0)+t(3, 2, 1)$ A1 or equivalent [3]

Alternative:

Direction of line $= \begin{pmatrix}1\\-2\\1\end{pmatrix} \times \begin{pmatrix}1\\-4\\5\end{pmatrix} = t\begin{pmatrix}3\\2\\1\end{pmatrix}$ M1A1

Find any point on line e.g. $\begin{pmatrix}-4\\-4\\0\end{pmatrix}, \begin{pmatrix}2\\0\\2\end{pmatrix}$ etc.

$\therefore \mathbf{r} = \begin{pmatrix}-4\\-4\\0\end{pmatrix} + t\begin{pmatrix}3\\2\\1\end{pmatrix}$ B1

Line l: $\mathbf{r} = (a, 2a+1, -3) + \alpha(3c, -3, c)$
Plane: $x - 2y + z = 4$

Distance A to plane:

$\left|\dfrac{a-2(2a+1)-3-4}{\sqrt{6}}\right| = \dfrac{15}{\sqrt{6}}$ M1

$3a + 9 = 15$ M1 correct use of modulus sign

$a = 2$ A1

$\sin\theta = \dfrac{3c+6+c}{\sqrt{6}\sqrt{9c^2+9+c^2}}$ M1A1

$\therefore \dfrac{4c+6}{\sqrt{6}\sqrt{9+10c^2}} = \dfrac{2}{\sqrt{6}}$ M1 solve for c

$6c^2 - 12c = 0$: $c = 2$ A1

(Penalise only once for negative values.) [7]

Q8.

6 (i)	Uses vector product to find vector perpendicular to both lines.	$\mathbf{n} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&3&0\\4&0&-1\end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - 12\mathbf{j}$	M1A1		
	Finds ***BA*** and its scalar product with unit perpendicular vector.	$\boldsymbol{BA} = \mathbf{i} + 10\mathbf{j} - 11\mathbf{j}$ perp.dist. $= \dfrac{-3\times1+4\times10+12\times11}{\sqrt{3^2+4^2+12^2}} = \dfrac{169}{\sqrt{169}} = 13$ $= 13$ (AG) (No penalty for sign errors made in **n**, which lead to correct result.)	M1A1	4	
(ii)		$\mathbf{p} = \begin{pmatrix}8+4\lambda\\8+3\lambda\\-7\end{pmatrix}$ $\mathbf{q} = \begin{pmatrix}7+4\mu\\-2\\4-\mu\end{pmatrix}$			
		$\mathbf{PQ} = \begin{pmatrix}-1-4\lambda+4\mu\\-10-3\lambda\\11-\mu\end{pmatrix} = t\begin{pmatrix}-3\\4\\-12\end{pmatrix}$	B1 M1A1		
		$\Rightarrow t = -1 \quad \lambda = -2 \quad \mu = -1$	M1		
		$\mathbf{p} = 2\mathbf{j} - 7\mathbf{k} \qquad \mathbf{q} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ (Award B1B1B1 if t assumed to be ±1 i.e. 3/5)	A1	5	**[9]**

6. Vectors

6(ii)	**Alternative Solution:**				
	Finds two parameter representation for **PQ**	$\mathbf{p} = \begin{pmatrix} 8+4\lambda \\ 8+3\lambda \\ -7 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 7+4\mu \\ -2 \\ 4-\mu \end{pmatrix}$			
	Uses scalar product between **PQ** and direction vector of at least one line and equates to zero	$\mathbf{PQ} = \begin{pmatrix} -1-4\lambda+4\mu \\ -10-3\lambda \\ 11-\mu \end{pmatrix}$	B1		
		$16\mu - 25\lambda = 34$	M1A1		
		$17\mu - 16\lambda = 15$	A1		
	Solves simultaneously.	$\mu = -1 \quad \lambda = -2$	M1A1		
	Obtains position vectors for P and Q	$\mathbf{p} = 2\mathbf{j} - 7\mathbf{k} \quad \mathbf{q} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	A1	**7**	
(i)	Obtains length of PQ.	$\sqrt{3^2 + (-4)^2 + 12^2} = 13$ (AG)	M1A1	**2**	**[9]**

Q9.

10	Uses vector product to find normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & 6 & 1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$	M1A1		
	Uses **r.n** = constant. Obtains cartesian equation of plane.	Equation of plane: $5x - 3y - 2z =$ constant $30 - 15 - 8 = 7$ $5x - 3y - 2z = 7$	M1 A1	4	
		Alternatively:			
		$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$	(M1)		
		$x = 6 + \lambda + 4\mu$ $y = 5 + \lambda + 6\mu$ $z = 4 + \lambda + \mu$	(A1)		
		Eliminates λ and μ.	(M1)		
		Obtains $5x - 3y - 2z = 7$	(A1)		

10 Contd.	Finds equation of perpendicular to plane through given point.	Equation of perpendicular: $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + t(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$	M1		
	Finds value of parameter at point in plane. Obtains foot of perpendicular.	$5(1+5t) - 3(10-3t) - 2(3-2t) = 7$ $\Rightarrow t = 1$ Foot of perpendicular is $6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$.	M1 A1 A1	4	
	Alternatively: Form sufficient equations, using orthogonality. Two will suffice if foot of perpendicular is expressed using parametric equation of plane.	Let foot of perpendicular be $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and using othogonality: $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow a + b + c = 14$ $\begin{pmatrix} a-1 \\ b-10 \\ c-3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix} = 0 \Rightarrow 4a + 6b + c = 67$ $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ lies in plane of l_1 and l_2 : $5a - 3b - 2c = 7$ $\Rightarrow 6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$	(M1A1) (M1A1)		
	Finds direction of common perpendicular.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -3 & 1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} - 5\mathbf{k}$	M1A1		
	Forms vector between known points on l_1 and l_3.	$\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$			
	Finds shortest distance by projection.	$\frac{1}{\sqrt{16+1+25}} \left\| \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} \right| = \frac{10}{\sqrt{42}} \quad (=1.54)$	M1A1 A1	5	**[13]**

10 Contd.	**Alternative for last part:**	Let P be on l_1 and Q be on l_3. $\mathbf{p} = \begin{pmatrix} 6+\lambda \\ 5+\lambda \\ 4+\lambda \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1+2v \\ 10-3v \\ 3+v \end{pmatrix}$ $\Rightarrow \overrightarrow{PQ} = \begin{pmatrix} -5-\lambda+2v \\ 5-\lambda-3v \\ -1-\lambda-3v \end{pmatrix}$	(M1)		
		Uses orthogonality conditions: $\Rightarrow \overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow -1-3\lambda = 0 \Rightarrow \lambda = -\frac{1}{3}$	(M1)		
		$\overrightarrow{PQ} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \Rightarrow -26+14v = 0 \Rightarrow v = \frac{13}{7}$	(A1)		
		$\Rightarrow \overrightarrow{PQ} = \frac{1}{21}\begin{pmatrix} -20 \\ -5 \\ 25 \end{pmatrix}$	(A1)		
		$\Rightarrow \left\lvert \overrightarrow{PQ} \right\rvert = \frac{5}{21}\sqrt{4^2+1^2+5^2} = \frac{5}{21}\sqrt{42}$	(A1)	(5)	

Q10.

2	Finds $\mathbf{a} \times \mathbf{b}$	$\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 4 & -3 & 2 \end{vmatrix} = \mathbf{i} - 10\mathbf{j} - 17\mathbf{k}$	M1 A1		
	Finds area of base.	$\frac{1}{2}\sqrt{1^2+(-10)^2+(-17)^2} = \frac{1}{2}\sqrt{390}$ $(=9.87)$	A1	3	
	Attempts to find height	Height $= \frac{(3i-j-k).(i-10j-17k)}{\sqrt{1^2+(-10)^2+(-17)^2}} = \frac{30}{\sqrt{390}}$ $(=1.519)$	M1		
	Finds volume	$\frac{1}{3} \times \frac{1}{2}\sqrt{390} \times \frac{30}{\sqrt{390}} = 5$	A1	2	**[5]**

6. Vectors

Q11.

9	Finds normal to plane.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -3 & 4 \end{vmatrix} = \mathbf{i} - 9\mathbf{j} - 7\mathbf{k}$	M1A1		
(i)	Deduces equation.	$\Pi:: \; x - 9y - 7z = \text{constant}$			
		Sub e.g. $(1, -1, 2) \Rightarrow$ constant $= -4$			
		$\Pi:: \; x - 9y - 7z = -4$	M1A1	4	
	General point on line inserted in plane equation to find λ.	$l: \; x = 6 + 2\lambda \quad y = -2 + \lambda \quad z = 1 - 4\lambda$			
		Sub in $\Pi \Rightarrow 6 + 2\lambda + 18 - 9\lambda - 7 + 28\lambda = -4$	M1		
		$\Rightarrow \lambda = -1$	A1		
		Position vector of intersection is **4i – 3j + 5k.**	A1	3	
(ii)	Distance of point from plane formula or triple scalar product method.	Either $\left\lvert \frac{6 + 18 - 7 + 4}{\sqrt{1 + 81 + 49}} \right\rvert$ Or $\frac{(2i + j - 4k).(i - 9j - 7k)}{\sqrt{1 + 81 + 49}}$	M1A1		
		$= \frac{21}{\sqrt{131}} \quad (=1.83)$	A1	3	
(iii)	Scalar product to find complement of angle.	$(2i + j - 4k).(i - 9j - 7k) = 21$	M1		
		$= \sqrt{4 + 1 + 16}\sqrt{1 + 81 + 49} \sin\theta$	A1		
		$\Rightarrow \sin\theta = \sqrt{\frac{21}{131}} \Rightarrow \theta = 23.6^0$ or 0.412 rad.	A1	3	**[13]**

6. Vectors

Q12.

11	**OR** Obtains direction of common perpendicular.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix} = -m\mathbf{i} + 4(1-m)\mathbf{j} + 4\mathbf{k}$	M1A1		
	Uses result for shortest distance between lines.	$\dfrac{\left\lvert \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \begin{pmatrix} -m \\ 4-4m \\ 4 \end{pmatrix} \right\rvert}{\sqrt{m^2 + 16(1-2m+m^2) + 16}} = 3$	M1A1		
	Solves equation.	$\Rightarrow \ldots \Rightarrow 19m^2 - 40m + 4 = 0$ $\Rightarrow (19m-2)(m-2) = 0$ $\Rightarrow m = 2$, since m is an integer. (AG)	A1 M1 A1	7	
	Finds relevant vectors.	$\mathbf{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\mathbf{AD} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$	B1		
	Use of cross-product.	$\dfrac{1}{\sqrt{17}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & -4 \end{vmatrix} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$	M1		
	Obtains shortest distance.	$\dfrac{1}{\sqrt{17}} \sqrt{4^2 + 1^2 + 1^2} = \sqrt{\dfrac{18}{17}}$ (= 1.03)	A1	3	
	Finds 2nd vector in BCD (**CD** may already have been found.)	$\mathbf{BC} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$	B1		
	Finds normal vector to BCD. (Normal to ACD already found.)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \sim 2\mathbf{i} + \mathbf{j} - \mathbf{k}$	M1		
	Finds angle between planes = angle between normal vectors.	$\cos\theta = \dfrac{(4\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{16+1+1}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{18}\sqrt{6}} = \dfrac{1}{\sqrt{3}}$	M1		
		$\therefore$ Angle between planes $= \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (AG)	A1	4	**[14]**

6. Vectors

11	**OR** **Alternatives for middle part:**				
	Or (a) Vector from D to any point on AC	$\begin{pmatrix}1+t\\-1\\-5-4t\end{pmatrix}$	(B1)		
	Uses orthogonality to obtain t.	$\begin{pmatrix}1+t\\-1\\-5-4t\end{pmatrix}\begin{pmatrix}1\\0\\-4\end{pmatrix}=0\Rightarrow t=-\frac{21}{17}$	(M1)		
	Finds magnitude of perpendicular.	$\frac{1}{\sqrt{17}}\sqrt{4^2+1^2+1^2}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	
	Or (b) Finds length of AD (or CD)	$\left\lvert\overrightarrow{AD}\right\rvert=\sqrt{27}$	(B1)		
	Finds projection of AD (or CD) onto AC.	$\left\lvert\frac{\begin{pmatrix}-1\\1\\5\end{pmatrix}\begin{pmatrix}1\\0\\-4\end{pmatrix}}{\sqrt{4^2+1^2}}\right\rvert=\frac{21}{\sqrt{17}}$	(M1)		
	Finds perpendicular by Pythagoras.	$\sqrt{27-\frac{441}{17}}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	

Q13.

9	Finds normal vector to plane.	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-2&-1\\1&2&-2\end{vmatrix}=\begin{pmatrix}6\\1\\4\end{pmatrix}$ or $\begin{matrix}x=2+\lambda+\mu\\y=-3-2\lambda+2\mu\\z=1-\lambda-2\mu\end{matrix}$	M1A1		
	Uses known point to find constant term.	Plane equation is $6x+y+4z=$ constant. Substitute $(2,-3,1)\Rightarrow 12-3+4=13$. Or eliminate λ and μ. $\Rightarrow \Pi_1: 6x+y+4z=13$	M1 A1	4	
	Angle between normals is equal to angle between planes.	$\cos\theta=\frac{(6\mathbf{i}+\mathbf{j}+4\mathbf{k}).(3\mathbf{i}-2\mathbf{j}-3\mathbf{k})}{\sqrt{6^2+1^2+4^2}\sqrt{3^2+2^2+3^2}}=\frac{4}{\sqrt{53}\sqrt{22}}$ $\Rightarrow\theta=83.3°$ or 1.45 rad.	M1A1 A1	3	
	Solve plane equation simultaneously.	$6x+y+4z=13$ and $3x-2y-3z=4$ Obtains e.g. $y+2z=1$ and $3x+z=6$ Or two of $(0,-11,6)$, $(11/6,0,1/2)$, $(2,1,0)$	M1 A1A1		
	(Note: may find direction from vector product and use with one point.)	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\-6\\3\end{pmatrix}$ (OE)	A1	4	
		Alternatively: Direction of line from vector product. Finds a point on line. States equation of line.	(M1A1) (A1) (A1)		**[11]**

6. Vectors

Q14.

9	Finds vector normal to Π.	$\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 2\\ 3 & 1 & -2\end{vmatrix}=2\mathbf{i}+8\mathbf{j}+7\mathbf{k}$	M1A1		
	Dot product of this with general point on l_1.	$\begin{pmatrix}3+8t\\6+5t\\12-8t\end{pmatrix}\cdot\begin{pmatrix}2\\8\\7\end{pmatrix}=138$ or $\begin{pmatrix}8\\5\\-8\end{pmatrix}\cdot\begin{pmatrix}2\\8\\7\end{pmatrix}=0$	A1√		
	Deduces result.	Independent of $t \Rightarrow$ parallel, or $\Rightarrow$ parallel.	A1	4	
	Cartesian equation of Π.	Π: $2x+8y+7z=21$	B1		
	Substitutes general point of l_2.	Sub. $x=5+2s$, $y=-4-s$, $z=7+s$	M1		
	Finds value of parameter.	$\Rightarrow s=-2$	A1		
	Finds p.v. of intersection.	and line meets Π at point with p.v. $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$	A1	4	
		Take (9,11,2) as A, (3,6,12) as B and let C be foot of perpendicular from A to l.			
		$AB=\sqrt{6^2+5^2+10^2}=\sqrt{161}$	B1		
	Finds distance from point to known point on l.	$BC=\dfrac{1}{\sqrt{(6^2\,5^2+8^2)(6\mathbf{i}+5\mathbf{j}-10\mathbf{k})(8\mathbf{i}+5\mathbf{j}-8\mathbf{k})}}=\dfrac{153}{\sqrt{153}}=\sqrt{153}$	M1 A1	4	**[12]**
	Finds distance along l from known point to foot of perpendicular from given point to l.	$AC=\sqrt{161-153}=\sqrt{8}$ or $2\sqrt{2}$ (= 2.83)	A1√		
		Alternatively:			
	F.t. on non-hypotenuse side (must be real).	$5+2s=2+\lambda+3\mu$ $-4-s=3-2\lambda+\mu$ $7+s=-1+2\lambda-2\mu$	(B1)		
	Writes a set of three equations in three unknowns for the intersection of l with Π.	$\Rightarrow s=-2$, $\Rightarrow \lambda=2$, $\mu=-1$	(M1A1)		
	Solves the set of equations.	and line meets Π at point with p.v. $\mathbf{i}-2\mathbf{j}+5\mathbf{k}$	(A1)		
	Finds p.v. of intersection.				

6. Vectors

Q15.

4	Finds vector product.	$\overrightarrow{AB} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}1\\1\\2\end{pmatrix}$	B1		
		$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 2 & 3\\1 & 1 & 2\end{vmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$	M1A1	3	
	Finds area of triangle.	Area of triangle $ABC = \frac{1}{2}\left\|\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\| = \frac{1}{2}\sqrt{3}$	M1A1		
	Finds length of perpendicular.	$\frac{1}{2}\sqrt{1^2+2^2+3^2}\,d = \frac{1}{2}\sqrt{3} \Rightarrow d = \sqrt{\frac{3}{14}}$	A1	3	**[6]**

Q16.

8	Finds normal to Π_1.	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 0 & 1\\1 & -1 & -2\end{vmatrix} = \mathbf{i}+3\mathbf{j}-\mathbf{k}$	M1A1		
	Finds Cartesian equation.	Equation of Π_1 : $x + 3y - z = 12$	A1	3	
	Finds angle between normals, using scalar product.	$\cos\theta = \left\|\frac{2-3-1}{\sqrt{11}\sqrt{6}}\right\|$	M1		
		$= \frac{2}{\sqrt{66}} \Rightarrow \theta = 75.7°$ or 1.32 rad.	A1	2	
	Finds direction of line of intersection, using vector product.	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\1 & 3 & -1\\2 & -1 & 1\end{vmatrix} = 2\mathbf{i}-3\mathbf{j}-7\mathbf{k}$	M1A1		
	Finds point common to both planes.	Point on both planes is e.g. (6,2,0)	M1A1		
	States vector equation.	$\mathbf{r} = 6\mathbf{i}+2\mathbf{j}+t(2\mathbf{i}-3\mathbf{j}-7\mathbf{k})$ (OE)	A1	5	**[10]**

11 O	Vector perpendicular to Π_1	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$	M1A1		
	Obtains cartesian equation	$3\times 2+1\times(-1)+(-2)\times(-6)=17$ $\Rightarrow 3x-y-6z=17$	M1 A1	4	
	Obtains area of triangle ABC.	$\frac{1}{2}\sqrt{3^2+1^2+6^2}=\frac{1}{2}\sqrt{46}$ (=3.39)	M1 A1		
	Obtains length of perpendicular from D to triangle ABC.	$\left\lvert\frac{9-6-12-17}{\sqrt{3^2+1^2+6^2}}\right\rvert=\frac{26}{\sqrt{46}}$	M1A1		
	Uses $\frac{1}{3}\times$ Base area $\times$ Height.	Either $\frac{1}{3}\times\frac{1}{2}\sqrt{46}\times\frac{26}{\sqrt{46}}=\frac{13}{3}$	M1A1		
	or triple scalar product method.	Or e.g. $\left\lvert\frac{1}{6}\begin{pmatrix}1\\5\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-6\end{pmatrix}\right\rvert=\frac{26}{6}=\frac{13}{3}$		6	
	Obtains normal to ABD.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & 4 \\ 2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix}$	M1A1		
	Uses scalar product	$\sqrt{3^2+1^2+6^2}\sqrt{5^2+7^2+10^2}\cos\theta=\begin{pmatrix}3\\-1\\-6\end{pmatrix}\cdot\begin{pmatrix}5\\7\\-10\end{pmatrix}$	M1		
	to find angle between normals and hence angle between Π_1 and Π_2.	$\Rightarrow\cos\theta=\frac{68}{\sqrt{46}\sqrt{174}}\Rightarrow\theta=40.5^\circ$	A1	4	**[14]**

6. Vectors

Q17.

11 E	Finds **PQ**.	$\mathbf{PQ} = \begin{pmatrix} -3+\mu-3\lambda \\ -6\mu-2\lambda \\ 12-2\mu+\lambda \end{pmatrix}$	M1A1		
	Finds direction of common perpendicular.	$\begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 1 & -6 & -2 \end{vmatrix} = \begin{pmatrix} -10 \\ 5 \\ -20 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$	M1		
	(Or uses scalar product of **PQ** with the direction vector of each line.)				
	Obtains two correct equations. e.g.	$-3+\mu-3\lambda = 12\mu+4\lambda$ $-24\mu-8\lambda = -12+2\mu-\lambda$	A1 A1		
	Solves.	$\mu = 1,\ \lambda = -2$	M1A1	8	
	Finds **p** and **q**.	$p = \begin{pmatrix} 4+3\lambda \\ 7+2\lambda \\ -1-\lambda \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ $q = \begin{pmatrix} 1+\mu \\ 7-6\mu \\ 11-2\mu \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$	A1		
	Finds common perpendicular.	$\mathbf{AB} \times \mathbf{PQ} = \begin{vmatrix} i & j & k \\ -1 & 0 & 4 \\ 2 & -1 & 4 \end{vmatrix} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}$	M1A1		
	Finds **PA** (or **QA** or **PB** or **QB**)	$\mathbf{PA}\begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix}$ $\mathbf{QA}\begin{pmatrix} 2 \\ 6 \\ -10 \end{pmatrix}$ $\mathbf{PB}\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}$ $\mathbf{QB}\begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix}$	B1		
	Uses triple scalar product to find shortest distance.	$\dfrac{\begin{pmatrix} 6 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}}{\sqrt{16+144+1}} = \dfrac{24+48-2}{\sqrt{161}} = \dfrac{70}{\sqrt{161}} = 5.52$	M1A1 A1	6	**[14]**
	Alternative for last 4 marks:	Plane through e.g. P in direction **PA :** $4x + 12y - 29 = 0$. Award M1A1. Then use of distance of point from line formula to get $\dfrac{70}{\sqrt{161}}$. Award M1A1.			

6. Vectors

Q18.

8	Finds normal to Π_1.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$	M1A1		
	Finds Cartesian equation.	Equation of Π_1 : $x + 3y - z = 12$	A1	3	
	Finds angle between normals, using scalar product.	$\cos\theta = \left\|\frac{2-3-1}{\sqrt{11}\sqrt{6}}\right\|$	M1		
		$= \frac{2}{\sqrt{66}} \Rightarrow \theta = 75.7°$ or 1.32 rad.	A1	2	
	Finds direction of line of intersection, using vector product.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$	M1A1		
	Finds point common to both planes.	Point on both planes is e.g. (6,2,0)	M1A1		
	States vector equation.	$\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + t(2\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$ (OE)	A1	5	**[10]**

Q19.

8	Finds normal	$\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$			
	and cartesian equation	$\overrightarrow{AB} \times \overrightarrow{AC} = -6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	M1A1		
		$6x + 2y - 5z = \text{constant} = 24 + 10 - 30 = 4$	M1A1	**(4)**	
	Finds where OD meets plane	Equation of OD: $\mathbf{r} = t(6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$	B1		
		$\Rightarrow 36t + 6t - 30t = 12t = 4 \Rightarrow t = \frac{1}{3}$	M1A1		
		E is the point (2,1,2).	A1	**(4)**	
	Obtains angles between planes	Using $(6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}).2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$			
		$\Rightarrow 12 + 2 - 10 = \sqrt{36 + 4 + 25}\sqrt{4 + 1 + 4}\sin\theta$	M1A1		
		$\Rightarrow \theta = 9.5°$ (0.166 rad).	A1	**(3)**	**[11]**

6. Vectors

Q20.

11	$\overrightarrow{AB}=\begin{pmatrix}6\\4\\-8\end{pmatrix}$ $\overrightarrow{CD}=\begin{pmatrix}5\\-2\\-4\end{pmatrix}$	B1
	Common perpendicular is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&-4\\5&-2&-4\end{vmatrix}=\begin{pmatrix}-16\\-8\\-16\end{pmatrix}\sim\begin{pmatrix}2\\1\\2\end{pmatrix}$	M1A1
	$\overrightarrow{AD}=\begin{pmatrix}1\\-4\\9\end{pmatrix}\Rightarrow$ shortest distance $=\dfrac{\left\lvert\begin{pmatrix}1\\-4\\9\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}\right\rvert}{\sqrt{4+1+4}}=\dfrac{16}{3}$ or 5.33	M1A1 [5]
	Normal to Π_1 is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&9\\3&2&-4\end{vmatrix}=\begin{pmatrix}-2\\31\\14\end{pmatrix}$	M1A1
	Normal to Π_2 is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&9\\5&-2&-4\end{vmatrix}=\begin{pmatrix}34\\49\\18\end{pmatrix}$	A1
	$\cos\theta=\dfrac{-2\times34+31\times49+14\times18}{\sqrt{2^2+31^2+14^2}\sqrt{34^2+49^2+18^2}}$	M1A1√
	$\Rightarrow\theta=36.7°$ (CAO)	A1 [6]

Q21.

	OR	
(i)	$\mathbf{r}=4\mathbf{i}-2\mathbf{j}+2\mathbf{k}+\lambda(\mathbf{i}+7\mathbf{j}+\mathbf{k})+\mu(3\mathbf{i}+\mathbf{j}-\mathbf{k})\Rightarrow A$ is in Π_1.	B1 [1]
(ii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&7&1\\3&1&-1\end{vmatrix}=\begin{pmatrix}-8\\4\\-20\end{pmatrix}\sim\begin{pmatrix}2\\-1\\5\end{pmatrix}$	M1A1 [2]
(iii)	L is $(12, -6, 6)$ $2x-y+5z=24+6+30=60$ $\mathbf{n}=t(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\Rightarrow 4t+t+25t=60\Rightarrow t=2$ $\mathbf{n}=4\mathbf{i}-2\mathbf{j}+10\mathbf{k}$	B1 B1 M1A1√ A1 [5]
(iv)	$\mathbf{m}=4\mathbf{i}-2\mathbf{j}+2\mathbf{k}+\dfrac{3}{4}(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=10\mathbf{i}-5\mathbf{j}+5\mathbf{k}$ (AG)	B1
	M is $(10, -5, 5)\Rightarrow\overrightarrow{NM}=6\mathbf{i}-3\mathbf{j}-5\mathbf{k}$	B1√
	$(6\mathbf{i}-3\mathbf{j}-5\mathbf{k})\times(2\mathbf{i}-\mathbf{j}+5\mathbf{k})=-\mathbf{i}+2\mathbf{j}$	M1A1
	Perpendicular distance is $\dfrac{\lvert 20(-\mathbf{i}+2\mathbf{j}\rvert}{\sqrt{30}}=\dfrac{20}{\sqrt{6}}=8.16$	M1A1 [6]
	(Mark various alternative methods in a similar manner.)	

6. Vectors

Q22.

10	(i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -3 \\ -2 & 1 & 3 \end{vmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \sim \begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{1} \end{pmatrix}$	M1A1
		$\overrightarrow{BA} = 6\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$	B1
		Shortest distance $\left\lvert \frac{\begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{1^2+1^2+1^2}} \right\rvert = \frac{4}{\sqrt{3}} \quad (= 2.31)$	M1A1 (5)
		Alternative	
		$\begin{pmatrix} -6-2\lambda-\mu \\ -4+\lambda+2\mu \\ 6+3\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \Rightarrow -13\lambda - 14\mu = 16$	(M1)
		$\begin{pmatrix} -6-2\lambda-\mu \\ -4+\lambda+2\mu \\ 6+3\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = 0 \Rightarrow 14\lambda + 113\mu = -26$	(A1)
		$\Rightarrow \lambda = -\frac{52}{9},\ \mu = \frac{38}{9}$	(M1A1)
		Shortest distance $= \frac{4}{\sqrt{3}}$. $\quad (= 2.31)$	(A1)
	(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 1 & 3 \end{vmatrix} = \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$	M1A1
		Cartesian equation of Π: $4x + 5y + z = -12 - 5 + 2 = -15$	M1A1 (4)
	(iii)	Distance of A from Π: $\left\lvert \frac{\begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}}{\sqrt{4^2+5^2+1^2}} \right\rvert$	M1A1
		$= \frac{38}{\sqrt{42}} \quad (= 5.86)$	A1 (3) **[12]**

6. Vectors

Q23.

110 (i)	$\mathbf{p} = \begin{pmatrix} 8+\lambda \\ 2-2\lambda \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 5 \\ 3+2\mu \\ -14-3\mu \end{pmatrix} \Rightarrow \overrightarrow{QP} = \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix}$	M1A1
	$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow 5\lambda + 4\mu = -5$	M1A1
	$\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} . \begin{pmatrix} 3+\lambda \\ -1-2\lambda-2\mu \\ 17+3\mu \end{pmatrix} = 0 \Rightarrow -4\lambda - 13\mu = 53$	A1
	Solving : $\lambda = 3, \mu = -5$	M1A1
	Whence: $\mathbf{p} = \begin{pmatrix} 11 \\ -4 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}$.	A1 (8)
(ii) (a)	$\overrightarrow{AP} = \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix}$, $\overrightarrow{AQ} = \begin{pmatrix} -3 \\ -9 \\ -2 \end{pmatrix}$	B1
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & 0 \\ -3 & -9 & -2 \end{vmatrix} = \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}$ (CAO) ; Area $= \frac{1}{2}\sqrt{2205}$ (=23.5)	M1A1 A1
(b)	$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} \Rightarrow$ Volume $= \frac{1}{3} \times \frac{1}{2}\sqrt{2205} \left\lvert \frac{\begin{pmatrix} -3 \\ 1 \\ -17 \end{pmatrix} . \begin{pmatrix} 12 \\ 6 \\ -45 \end{pmatrix}}{\sqrt{2205}} \right\rvert = \frac{735}{6} (= 122.5)$	M1A1 (6) **Total: 14**

	Alternative: for marks 3,4,5,6 and 7 in part **(i)** $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{vmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ $6k - \lambda = 3$ $3k + 2\lambda + 2 = -1$ $2k - 3\mu = 17$ Solving; $k = 1$, $\lambda = 3$ and $\mu = -5$ (If k missing, or assumed to be 1, deduct 1 mark.)	(M1A1) (A1) (M1A1)

Q24.

8	$\begin{pmatrix} 3+2\lambda \\ \lambda \\ 2-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 3$ $3+2\lambda+2\lambda+2-2\lambda = 3 \Rightarrow \lambda = -1$ P is $(1, -1, 4)$ (Accept position vector.) $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \begin{pmatrix} -5 \\ 4 \\ -3 \end{pmatrix}$ Direction of PQ is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 4 & -3 \\ 1 & 2 & 1 \end{vmatrix} = \begin{pmatrix} 10 \\ 2 \\ -14 \end{pmatrix} \sim \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$ (OE) Equation of PQ is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$ [N.B. For methods not using **w** at most three B1 marks.]	M1 A1 A1 (3) B1 M1A1 (3) M1A1 dM1A1 (4) **Total 10**

6. Vectors

Q25.

11 E	E.g. $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$	**M1A1**
	$\dfrac{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}{\sqrt{4^2+2^2+1^2}} = \dfrac{4}{\sqrt{21}}$ (AG)	**M1A1** [4]
	$\mathbf{p} = \dfrac{3}{\sqrt{21}}\left(\dfrac{4\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{21}}\right) = \dfrac{1}{7}(4\mathbf{i}+2\mathbf{j}+\mathbf{k})$	**B1**
	Line AP: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$	**M1A1**
	For Q $1-3t=0 \Rightarrow t = \dfrac{1}{3} \Rightarrow \mathbf{q} = \dfrac{1}{3}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$	**M1A1** [5]
	E.g. $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{BQ} = \dfrac{1}{3}\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$	**B1**
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & -4 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$	**M1A1**
	$\cos^{-1}\left\lvert\dfrac{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}}{\sqrt{21}.\sqrt{21}}\right\rvert = \cos^{-1}\dfrac{8+2+4}{21} = \cos^{-1}\dfrac{14}{21} = \cos^{-1}\dfrac{2}{3}$ (AG)	**M1A1** [5] **Total 14**

6. Vectors

Q26.

8	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -8 \\ -3 & 4 & -5 \end{vmatrix} = \begin{pmatrix} 17 \\ 34 \\ 17 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$	**M1 A1**
	So $x+2y+z = \text{const} \Rightarrow \text{const} = 2-2+3 = 3$ (using a point) $\Rightarrow$ $x+2y+z=3$	**M1** **A1** [4]
	$\sqrt{9+1+4}\sqrt{1+4+1}\cos\theta = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} . \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \Rightarrow \cos\theta = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{\sqrt{84}}$	**M1 M1**
	$\Rightarrow \theta = 70.9°$ or 1.24 radians	**A1** [3]
	Direction of line of intersection is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$	**M1A1**
	Finds point common to both planes is $(-1,0,4)$ or $\left(\frac{13}{7},\frac{4}{7},0\right)$ or $(0,\frac{1}{5},\frac{13}{5})$	**M1**
	Equation of line of intersection is eg $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ 1 \\ -7 \end{pmatrix}$.	**A1**√ [4]

Q27.

11 (o) (i)	Direction perpendicular to AB and CD: $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 4 & 3-\lambda & 1 \end{vmatrix} = \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}$	**M1A1**
	$\overrightarrow{DB} = \begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix}$. Hence $\left\lvert \frac{\begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}}{\sqrt{(\lambda - 2)^2 + 16 + 16}} \right\rvert = 3$ or equivalent	**M1A1**
(ii)	$\Rightarrow (5\lambda - 2)^2 = 9(\lambda^2 - 4\lambda + 36)$ $\Rightarrow \ldots \Rightarrow \lambda^2 + \lambda - 20 = 0$ **(AG)**	**M1M1 A1** [7]
	$(\lambda + 5)(\lambda - 4) = 0 \Rightarrow \lambda = -5, 4$	**B1**
	$\lambda = 4 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -1 & 3 & 7 \end{vmatrix} = \begin{pmatrix} -10 \\ -29 \\ 11 \end{pmatrix}$	**M1A1**
	$\lambda = -5 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & 1 \\ -1 & 12 & 7 \end{vmatrix} = \begin{pmatrix} 44 \\ -29 \\ 56 \end{pmatrix}$	**A1**
	$\cos\theta = \frac{-440 + 841 + 616}{\sqrt{10^2 + 29^2 + 11^2}\sqrt{44^2 + 29^2 + 56^2}} = \frac{1017}{\sqrt{1062}\sqrt{5913}}$	**M1A1**
	$\Rightarrow \theta = 66.1°$	**A1** [7]

6. Vectors

Q28.

11 E	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 3 & -4 & -2 \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	**M1A1**	
	Obtains three equations. E.g. from $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ $t - 3s + 2k = 4$ $-3t + 4s - k = -2$ $-t + 2s + 5k = 4$	**M1A1**	
	Solves: $s = -1$, $t = -1$, $k = 1$	**M1A1**	
	$\mathbf{p} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ (Both required)	**A1**	[7]
	ALT METHOD: Find PQ in terms of s,t **M1** Calculate scalar products direction vectors for l_1 and l_2 **M1,A1,A1** Solve simultaneous equations **M1 A1** p and q **A1**		
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -2 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix}$	**B1**	
	$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	**B1B1**	[3]

	Direction of l_3 is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$	**B1**	
	Normal to Π is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 22 & 19 & -5 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} 90 \\ -120 \\ -60 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$	**M1**	
	Cartesian equation of Π is $3x - 4y - 2z = 1$ (Since P lies in Π)	**A1**	
	Vector equation of l_3 is $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ (OE) (Since $z = 0$.)	**B1**	[4]

6. Vectors

Q29.

12O(i)	$\overrightarrow{AC}=\begin{pmatrix}2\\-2\\-2\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$ $\overrightarrow{CD}=\begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$	**B1**	May find $\overrightarrow{AD}, \overrightarrow{BC}$ *or* $\overrightarrow{BD}$ instead of AC
	$\overrightarrow{AB}\times\overrightarrow{CD}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\2&-4&\alpha-1\end{vmatrix}=\begin{pmatrix}5-\alpha\\3-\alpha\\-2\end{pmatrix}\sim\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}$	**M1A1**	
	$\left\lvert\dfrac{\begin{pmatrix}2\\-2\\-2\end{pmatrix}\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}}{\sqrt{(\alpha-5)^2+(\alpha-3)^2+4}}\right\rvert=2\sqrt{2}\Rightarrow 2\sqrt{2}\sqrt{2\alpha^2-16\alpha+38}=8$	**M1A1**	Substitutes their vectors into correct formula
	$\Rightarrow 2\alpha^2-16\alpha+38=8\Rightarrow\alpha^2-8\alpha+15=0$	**A1**	
	$\Rightarrow(\alpha-3)(\alpha-5)=0\Rightarrow\alpha=3$ or 5 (AG)	**A1**	
	Total:	**7**	

12O(ii)	$\overrightarrow{AD}=\begin{pmatrix}4\\-6\\0\end{pmatrix}$ $\left(or\ \overrightarrow{CD}=\begin{pmatrix}2\\-4\\2\end{pmatrix}\right)$	**B1**	*Alt method:* Let P be point on AC with parameter $\boldsymbol{\lambda}$ **M1**
	Distance of D from $AC=\dfrac{1}{\sqrt{1+1+1}}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-6&0\\1&-1&-1\end{vmatrix}=\sqrt{\frac{56}{3}}=4.32$ (or with CD)	**M1A1**	Use DP.AC = 0 to find $\boldsymbol{\lambda}$ (= -5/3) **M1** Find length **A1**
	Total:	**3**	
12O(iii)	ABC: $\mathbf{n}_1=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&-1\\1&-1&1\end{vmatrix}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ ABD: $\mathbf{n}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\2&-3&0\end{vmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$	**B1B1**	
	$\cos\theta=\left\lvert\frac{-3-2+0}{\sqrt{2}\sqrt{14}}\right\rvert\Rightarrow\theta=19.1°$ *or* 0.333 *rads*	**M1A1**	
	Total:	**4**	

6. Vectors

Q30.

9(i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{pmatrix} -7 \\ -1 \\ 3 \end{pmatrix}$	**M1A1**	
	Cartesian equation of Π_1 is $7x + y - 3z = \text{const}$	**M1**	
	$7\times1+2-3\times1=6 \Rightarrow 7x+y-3z=6$	**A1**	
	Total:	**4**	
9(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}$	**M1A1**	
	$5\times2-(-3)+7\times1=20 \Rightarrow 5x-y+7z=20$	**M1A1**	
	Total:	**4**	
9(iii)	$\cos\theta = \left\lvert \dfrac{\begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix} . \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}}{\sqrt{49+1+9}\sqrt{25+1+49}} \right\rvert$	**M1M1**	
	$\Rightarrow \cos\theta = \dfrac{13}{\sqrt{59}\sqrt{75}} \Rightarrow \theta = 78.7°$ or 1.37 rad.	**A1**	
	Total:	**3**	

Q31.

6(i)	$\overrightarrow{AB} = \mathbf{i}+5\mathbf{j}-2\mathbf{k}$ $\overrightarrow{BC} = -4\mathbf{i}-2\mathbf{j}+5\mathbf{k}$ $\overrightarrow{AC} = -3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$	**B1**	2 correct required
	$\overrightarrow{AB}\times\overrightarrow{BC} = 21\mathbf{i}+3\mathbf{j}+18\mathbf{k}$ (*)	**M1A1**	OE
	Area of triangle $ABC = \frac{1}{2}\sqrt{21^2+3^2+18^2} = 13.9\left(\frac{3}{2}\sqrt{86}\right)$	**A1**	
	Alt method: Use scalar product to find angle	**(M1A1**	
	Find area using Area = ½ *ab* sin *C* or equivalent	**M1A1)**	
		4	
6(ii)	$d = \dfrac{\lvert\overrightarrow{AB}\times\overrightarrow{BC}\rvert}{\lvert\overrightarrow{BC}\rvert} = \dfrac{\sqrt{21^2+3^2+18^2}}{\sqrt{4^2+2^2+5^2}}$	**M1A1**	Alt method: Find angle at *C*
	$= 4.15\left(\frac{1}{5}\sqrt{430}\right)$	**A1**	Area triangle = sin *C* × |*AC*|
	Alt method: Use equation of BC to find D (foot of perpendicular) in terms of parameter **and** scalar product to find parameter , λ= 8/15. Find length	**(M1A1)**	
		3	
6(iii)	From (*) Cartesian equation is $7x + y + 6z = \text{const}$.	**M1**	
	Through (2, –1 , 1) Hence $7x + y + 6z = 19$	**A1**	
		2	

6. Vectors

Q32.

10(i)(a)	Point on l_1 is $\begin{pmatrix}\lambda a\\9-\lambda\\2+\lambda\end{pmatrix}$ and on l_2 is $\begin{pmatrix}-6-\mu a\\-5+2\mu\\10+4\mu\end{pmatrix}$	**B1**	
	Point of intersection: $\begin{pmatrix}\lambda a\\9-\lambda\\2+\lambda\end{pmatrix}=\begin{pmatrix}-6-\mu a\\-5+2\mu\\10+4\mu\end{pmatrix}$	**M1**	Equates coordinates of points
	$\Rightarrow \mu=1,\ \lambda=12,\ a=-\frac{6}{13}$	**A1**	AG
		3	

10(i)(b)	Normal to the plane: $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\-\frac{6}{13} & -1 & 1\\\frac{6}{13} & 2 & 4\end{vmatrix}=-6\mathbf{i}+\frac{30}{13}\mathbf{j}-\frac{6}{13}\mathbf{k}\ \sim -13\mathbf{i}+5\mathbf{j}-\mathbf{k}$	**M1 A1**	Uses cross product to find normal to the plane AEF
	Using point on plane: e.g. $5(9)-2=43$	**M1**	Substitutes a point
	Equation of plane: $-13x+5y-z=43$	**A1**	AEF
		4	

10(ii)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\a & -1 & 1\\-a & 2 & 4\end{vmatrix}=-6i-5aj+ak$ $\left\|\begin{pmatrix}-6\\-5a\\a\end{pmatrix}\right\|=\sqrt{36+26a^2}$	**B1**	Finds the magnitude of the cross product of the direction vectors of the lines
	$\begin{pmatrix}-6\\-5-9\\10-2\end{pmatrix}\cdot\begin{pmatrix}-6\\-5a\\a\end{pmatrix}=36+78a$	**M1 A1**	Takes dot product of the correct vectors
	$3\sqrt{30}\sqrt{36+26a^2}=\lvert 36+78a\rvert$	**M1**	Puts distance equal to $3\sqrt{30}$
	$\Rightarrow 15\left(18+13a^2\right)=\left(6+13a\right)^2$ $\Rightarrow 26(a-3)^2=0 \qquad \Rightarrow a=3$	**A1**	
		5	

6. Vectors

Q33.

7(i)	Solve two equations $9+2\lambda=7+2\mu$ $13+3\lambda=-2-3\mu$	**M1**	
	to obtain $\lambda=-3$ and $\mu=-2$.	**A1**	
	Use third equation to obtain $a=2$.	**A1**	
7(ii)	Normal to the plane is $\mathbf{n}=-12\mathbf{i}+4\mathbf{k}$	**M1 A1**	
	Perpendicular distance $=\frac{1}{\sqrt{160}}\begin{pmatrix}-6\\6\\-8\end{pmatrix}\cdot\begin{pmatrix}-12\\0\\4\end{pmatrix}=\sqrt{10}=3.16$ **Alt method:** Finds equation of plane M1, A1: Finds foot of perpendicular from P to plane M1 Hence length A1 **Alt method:** Find equation of plane 3x – z + 7 = 0 M1, A1 Use formula $\frac{\lvert 3\times3+0\times1-6+7\rvert}{\sqrt{9+1}}=\sqrt{10}$ M1, A1	**M1 A1**	
7(iii)	Cross product of direction of $\boldsymbol{P}$ to l_2 with direction of l_2 $\begin{pmatrix}-6\\6\\-8\end{pmatrix}\times\begin{pmatrix}-1\\2\\-3\end{pmatrix}=\begin{pmatrix}-2\\-10\\-6\end{pmatrix}$	**M1 A1**	**Alt method:** Find N (foot of perpendicular) in terms of parameter and uses scalar product with n to find parameter M1, A1 so PN = $\begin{pmatrix}3\\0\\-1\end{pmatrix}$
	Perpendicular distance from $\boldsymbol{P}$ to l_2 is $\frac{\lvert 2\mathbf{i}+10\mathbf{j}+6\mathbf{k}\rvert}{\lvert -\mathbf{i}+2\mathbf{j}-3\mathbf{k}\rvert}=\sqrt{10}=3.16$	**M1 A1**	Find length PN M1, A1
		11	

Q34.

8(i)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-4&1&3\\0&1&2\end{vmatrix}=-\mathbf{i}+8\mathbf{j}-4\mathbf{k}$	**M1**	Finds normal to $\boldsymbol{\Pi}_1$.
	$-x+8y-4z=3$	**M1 A1**	Uses point on plane to find cartesian equation, AEF.
		3	
8(ii)	$\begin{pmatrix}-1\\8\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=9\sqrt{11}\cos\theta$	**M1**	Uses scalar product.
	$\Rightarrow\theta=72.5°$	**A1**	Accept 1.26 rad.
		2	
8(iii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&1&-1\\-1&8&-4\end{vmatrix}=4\mathbf{i}+13\mathbf{j}+25\mathbf{k}$	**M1 A1**	Finds direction of line of intersection.
	Point on both planes is, e.g. $(1,1,1)$.	**M1 A1**	Finds point common to both planes.
	$\mathbf{r}=\mathbf{i}+\mathbf{j}+\mathbf{k}+\lambda(4\mathbf{i}+13\mathbf{j}+25\mathbf{k})$	**A1**	States vector equation of line.
		5	

6. Vectors

Q35.

10(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}1\\1.5\\1\end{pmatrix}=\frac{1}{2}\overrightarrow{AB}$	**B1**	Or shows if parallel, then m=3/2
		1	
10(ii)	$\overrightarrow{AB}=\begin{pmatrix}2\\3\\2\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}1\\m\\1\end{pmatrix}$ **and** $\overrightarrow{AC}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ **or** AD $=\begin{pmatrix}-1\\m-1\\1\end{pmatrix}$ **or** BC$=\begin{pmatrix}-4\\-4\\-2\end{pmatrix}$	**B1**	
	$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\1&m&1\end{vmatrix}=\begin{pmatrix}3-2m\\0\\2m-3\end{pmatrix}$ (so parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$)	**M1A1**	Finds common perpendicular using cross product.
	$\frac{\lvert AC.\boldsymbol{n}\rvert}{\lvert\boldsymbol{n}\rvert}=\frac{\lvert-2(3-2m)+0+0\rvert}{\sqrt{(3-2m)^2+(2m-3)^2}}$ o.e.	**M1**	Uses formula for shortest distance.
	$=\frac{2}{\sqrt{2}}=\sqrt{2}$	**A1**	
		5	
10(iii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-2&-1&0\end{vmatrix}=\begin{pmatrix}2\\-4\\4\end{pmatrix}\sim\begin{pmatrix}1\\-2\\2\end{pmatrix}$ o.e.	**M1A1**	Finds normal to plane ABC (AEF).
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&3&2\\-1&1&1\end{vmatrix}=\begin{pmatrix}1\\-4\\5\end{pmatrix}$ o.e.	**A1**	Finds normal to ABD (AEF).
	$\cos\theta=\frac{1+8+10}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+4^2+5^2}}\left(=\frac{19}{\sqrt{378}}\right)$	**M1A1FT**	Uses formula for angle between two lines.
	$\Rightarrow\theta=12.2°$	**A1**	CAO.
		6	

6. Vectors

Q36.

3	$\overrightarrow{OP}=\begin{pmatrix}6+\lambda\\2+\lambda\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\4-6\mu\\\mu\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=0$ **Or** $\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}=k\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&-6&1\end{vmatrix}=k\begin{pmatrix}1\\-1\\-6\end{pmatrix}$	**M1**	Uses that dot product of $\overrightarrow{PQ}$ with line directions is 0. Or, alternatively, $\overrightarrow{PQ}$ is multiple of common perpendicular.
	$-2\lambda-6\mu=0$	**A1**	Deduces one equation. CWO.
	$\begin{pmatrix}-2-\lambda\\2-\lambda-6\mu\\-7+\mu\end{pmatrix}\begin{pmatrix}0\\-6\\1\end{pmatrix}=0\Rightarrow6\lambda+37\mu=19$	**A1**	Deduces second equation. CWO.
	$\lambda=-3,\ \mu=1$	**M1 A1**	Solves simultaneous equations.
	$\overrightarrow{OP}=\begin{pmatrix}3\\-1\\7\end{pmatrix},\overrightarrow{OQ}=\begin{pmatrix}4\\-2\\1\end{pmatrix}$	**A1**	States $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
		8	

Q37.

7(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\4\\5\end{pmatrix},\ \overrightarrow{CD}=\begin{pmatrix}1\\1\\0\end{pmatrix}$	**B1**	Find the directions of the lines.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\1&1&0\end{vmatrix}=\begin{pmatrix}-5\\5\\-2\end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular. Allow any non-zero scalar multiple.
	$\frac{\begin{pmatrix}1\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\5\\-2\end{pmatrix}}{\sqrt{5^2+5^2+2^2}}=\frac{18}{\sqrt{54}}=\sqrt{6}=2.45$	**M1 A1**	Uses formula for shortest distance.
		5	
7(ii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\2&6&1\end{vmatrix}=\begin{pmatrix}-26\\8\\4\end{pmatrix}\sim\begin{pmatrix}-13\\4\\2\end{pmatrix}$	**M1 A1**	Finds normal to the plane.
	$\frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\4\\2\end{pmatrix}}{\sqrt{1^2+1^2+0^2}\sqrt{13^2+4^2+2^2}}=-\frac{9}{\sqrt{378}}=-\frac{\sqrt{42}}{14}$	**M1 A1**	**FT** Uses dot product of their normal with direction of line.
	$\cos^{-1}\left(-\frac{\sqrt{42}}{14}\right)-90=27.6°\quad(\text{or }0.481\text{rad})$	**A1**	
		5	

6. Vectors

Q38.

6(i)	$\overline{AB} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$	**B1**	
	$\overline{OC} \times \overline{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 2 & 7 \end{vmatrix} = \begin{pmatrix} -9 \\ -6 \\ 3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular.
	$\frac{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}{\sqrt{3^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{14}} = 0.267$	**M1 A1**	Uses formula for shortest distance.
		5	
6(ii)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = t \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$	**M1 A1**	Finds normal to plane.
	$-(0) + 4(0) + 5(0) = 0$	**M1**	Uses point on plane.
	$-x + 4y + 5z = 0$	**A1**	AEF
		4	

Q39.

5(a)	$\begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -9 \end{pmatrix}$	**B1**
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 4 \\ 0 & 5 & 6 \end{vmatrix} = \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix}$	**M1 A1**
	$\frac{1}{\sqrt{77}} \left\| \begin{pmatrix} 0 \\ -5 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix} \right\| = \frac{15}{\sqrt{77}} = 1.71$	**M1 A1**
		5
5(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 4 \\ 1 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$	**M1 A1**
	$4(3) + 3(0) - 4(3) = 0 \Rightarrow 4x + 3y - 4z = 0$	**M1 A1**
		4
5(c)	$\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} = \sqrt{61}\sqrt{41} \cos\alpha \Rightarrow \cos\alpha = \frac{-9}{\sqrt{61}\sqrt{41}}$ **(A1 FT** *their* normal)	**M1 A1FT**
	Acute angle between l_2 and Π is $\alpha - 90 = 10.4^\circ$	**A1**
		3

6. Vectors

Q40.

7(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$	**M1 A1**
	$-5(-5) = 25$	**M1**
	$-2x - 5y + 5z = 25$	**A1**
		4
7(b)	$\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix}$	**M1**
	$\frac{1}{\sqrt{54}}\left\lvert \begin{pmatrix} -4 \\ -7 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix} \right\rvert = \frac{53}{\sqrt{54}} = 7.21$	**M1 A1**
		3

7(c)	$\overrightarrow{OP} = \begin{pmatrix} 5\lambda \\ -5 \\ 2\lambda \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 4 \\ 2+\mu \\ -2+\mu \end{pmatrix} \Rightarrow \overrightarrow{PQ} = \begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix}$	**M1 A1**
	$\begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = 0$ or $\begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix} = k\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$	**M1**
	$-29\lambda + 2\mu = -16$	**A1**
	$\begin{pmatrix} 4-5\lambda \\ 7+\mu \\ -2+\mu-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow -2\lambda + 2\mu = -5$	**A1**
	$\lambda = \frac{11}{27} \Rightarrow \overrightarrow{OP} = \frac{1}{27}\begin{pmatrix} 55 \\ -135 \\ 22 \end{pmatrix} = \frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$	**M1 A1**
	$\mathbf{r} = \frac{1}{27}\begin{pmatrix} 55 \\ -135 \\ 22 \end{pmatrix} + k\begin{pmatrix} -2 \\ -5 \\ 5 \end{pmatrix}$ (**B1 FT** their common perpendicular)	**B1 FT**
		8

6. Vectors

Q41.

4(a)	$\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ $\overrightarrow{AC} = 3\mathbf{i} - \mathbf{j}$	**B1**	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 3 & -1 & 0 \end{vmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$	**M1 A1**	Finds normal to the plane ABC.
	$-2(-1) - 6(1) + 7(2) = 10 \Rightarrow -2x - 6y + 7z = 10$	**M1 A1**	Substitutes point.
	Alternative method for question 4(a)		
	Setting up 3 equations using points given.	**M1**	
	$-2x - 6y + 7z = 10$	**A1 A1 A1 A1**	OE
		5	
4(b)	$\frac{10}{\sqrt{2^2 + 6^2 + 7^2}} = \frac{10}{\sqrt{89}}$ OE	**M1 A1**	Divides by magnitude of normal vector. 1.06…
		2	

4(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$	**M1 A1**	Finds normal to the plane OAB.
	$\begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \sqrt{89}\sqrt{29}\cos\theta$	**M1**	Uses dot product correctly.
	36.2°	**A1**	
		4	

Q42.

7(a)	$\overrightarrow{AB} = -\mathbf{j} + 3\mathbf{k}$ $\overrightarrow{AC} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$	**B1**	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$	**M1 A1**	Finds normal to the plane ABC. OE
	$5(0) + 3(-2) + 1(1) = -5 \Rightarrow 5x + 3y + z = -5$	**M1 A1**	Substitutes point. OE
		5	
7(b)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$	**M1 A1**	Finds normal to the plane OBC.
	$\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = \sqrt{35}\sqrt{45}\cos\theta$	**M1**	Uses dot product of normal vectors.
	$28.1°$	**A1**	0.49(0) rad
		4	

7(c)	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & -1 & 3\\ t & 1 & -1\end{vmatrix} = \begin{pmatrix}-2\\3t\\t\end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\left\lvert \frac{1}{\sqrt{4+10t^2}}\begin{pmatrix}2\\-4\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\3t\\t\end{pmatrix}\right\rvert = \left\lvert\frac{-4-10t}{\sqrt{4+10t^2}}\right\rvert$	**M1 A1**	Uses formula for perpendicular distance.
	$\left\lvert\frac{-4-10t}{\sqrt{4+10t^2}}\right\rvert = \sqrt{10} \Rightarrow (4+10t)^2 = 10\left(4+10t^2\right)$	**M1**	Sets equal to $\sqrt{10}$ and solves for t.
	$t = \frac{3}{10}$	**A1**	
		6	

Q43.

11 E	E.g. $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 0\\ -1 & 0 & 4\end{vmatrix} = \begin{pmatrix}8\\4\\2\end{pmatrix} \sim \begin{pmatrix}4\\2\\1\end{pmatrix}$	2	**M1A1**	
	$\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}4\\2\\1\end{pmatrix}}{\sqrt{4^2+2^2+1^2}} = \frac{4}{\sqrt{21}}$ (AG)	2	**M1A1**	
		4		
	$\mathbf{p} = \frac{3}{\sqrt{21}}\left(\frac{4\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{21}}\right) = \frac{1}{7}(4\mathbf{i}+2\mathbf{j}+\mathbf{k})$	1	**B1**	
	Line AP: $\mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-3\\2\\1\end{pmatrix}$	2	**M1A1**	
	For Q $1-3t=0 \Rightarrow t = \frac{1}{3} \Rightarrow \mathbf{q} = \frac{1}{3}\begin{pmatrix}0\\2\\1\end{pmatrix}$	2	**M1A1**	
		5		

	E.g. $\overrightarrow{AB} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\overrightarrow{BQ} = \frac{1}{3}\begin{pmatrix}0\\-4\\1\end{pmatrix}$	1	**B1**	
	$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 0\\ 0 & -4 & 1\end{vmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix}$	2	**M1A1**	
	$\cos^{-1}\left\lvert\frac{\begin{pmatrix}4\\2\\1\end{pmatrix}\begin{pmatrix}2\\1\\4\end{pmatrix}}{\sqrt{21}.\sqrt{21}}\right\rvert = \cos^{-1}\frac{8+2+4}{21} = \cos^{-1}\frac{14}{21} = \cos^{-1}\frac{2}{3}$ (AG)	2	**M1A1**	
		5		

6. Vectors

Q44.

6(a)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix} = -m\mathbf{i} - 4(m-1)\mathbf{j} + 4\mathbf{k}$	2	**M1A1**	Finds direction of common perpendicular
	$\dfrac{\left\| \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} -m \\ 4-4m \\ 4 \end{pmatrix} \right\|}{\sqrt{(m^2 + 16(1-2m+m^2)+16)}} = 3$	2	**M1A1**	Uses result for shortest distance between lines
	$19m^2 - 40m + 4 = 0$	2	**M1A1**	
	$(19m-2)(m-2) = 0 \quad m = 2$, since m is an integer	1	**A1**	AG Dependent on method seen for solving quadratic equation
		7		
6(b)	$\overrightarrow{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\overrightarrow{AD} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$	1	**B1**	Finds relevant vectors
	$\dfrac{1}{\sqrt{17}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & -4 \end{vmatrix} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$	1	**M1**	Uses cross-product The unit vector parallel to $\overrightarrow{CA}$ is required
	$\dfrac{1}{\sqrt{17}} \sqrt{4^2 + 1^2 + 1^2} = \sqrt{\dfrac{18}{17}} (= 1.03)$	1	**A1**	
		3		

6(c)	$\overrightarrow{BC} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$	1	**B1**	Finds second vector in BCD
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$	1	**M1**	Finds normal vector to BCD
	$\cos\theta = \dfrac{(4\mathbf{i} - \mathbf{j} + \mathbf{k}) \bullet (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{16+1+1}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{18}\sqrt{6}} = \dfrac{1}{\sqrt{3}}$	1	**M1**	Finds angle between normal vectors
	Angle between planes $= \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$	1	**A1**	AG
		4		

6. Vectors

Q45.

6(a)	$\begin{pmatrix}0\\1\\t\end{pmatrix}-\begin{pmatrix}t\\1\\0\end{pmatrix}=t\begin{pmatrix}-1\\0\\1\end{pmatrix}$	**B1**	Finds vector from any point on l_1 to any point on l_2.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&-1&0\\0&-2&1\end{vmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\frac{t}{\sqrt{21}}\left\lvert\begin{pmatrix}-1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\4\end{pmatrix}\right\rvert=\sqrt{21}\Rightarrow t=\frac{21}{5}=4.2$	**M1 A1**	Uses formula for shortest distance.
		5	
6(b)	$\mathbf{r}=\frac{21}{5}\mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j})+\mu(-2\mathbf{j}+\mathbf{k})$	**B1 FT**	Using their value of t.
		1	
6(c)	$\begin{pmatrix}0\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}5\\-6\\7\end{pmatrix}=\sqrt{5}\sqrt{110}\cos\alpha$ leading to $\cos\alpha=\frac{19}{\sqrt{5}\sqrt{110}}$	**M1 A1**	Uses dot product of $-2\mathbf{j}+\mathbf{k}$ and normal.
	Acute angle between l_2 and Π_2 is $90-\alpha=54.1^\circ$	**A1**	
		3	

Q46.

6(a)	$-(-2\mathbf{i}-2\mathbf{j}+4\mathbf{k})+(-\mathbf{i}-2\mathbf{j}+\mathbf{k})=\mathbf{i}-3\mathbf{k}$	**B1**	OE. Finds direction vector from P to a point of l_1.
	$\mathbf{r}=-2\mathbf{i}-2\mathbf{j}+4\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j})+\mu(\mathbf{i}-3\mathbf{k})$ $\boldsymbol{r}=-\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}+\lambda(2\boldsymbol{i}-3\boldsymbol{j})+\mu(\boldsymbol{i}-3\boldsymbol{k})$	**B1 FT**	OE. FT *their* $\mathbf{i}-3\mathbf{k}$
		2	

6(b)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&0\\3&-1&3\end{vmatrix}=\begin{pmatrix}-9\\-6\\7\end{pmatrix}$	**M1 A1**	OE. Finds vector perpendicular to Π_2.
	$-9(3)-6(0)+7(-2)=-41$	**M1**	Uses point on Π_2.
	$9x+6y-7z=41$	**A1**	
		4	
6(c)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&0\\1&0&-3\end{vmatrix}=\begin{pmatrix}9\\6\\3\end{pmatrix}\sim\begin{pmatrix}3\\2\\1\end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π_1.
	$\begin{pmatrix}3\\2\\1\end{pmatrix}\bullet\begin{pmatrix}9\\6\\-7\end{pmatrix}=\sqrt{14}\sqrt{166}\cos\alpha$ leading to $\cos\alpha=\frac{32}{\sqrt{14}\sqrt{166}}$	**DM1** **A1 FT**	Dot product using their normal vectors.
	48.4°	**A1**	0.845 rad
		5	

6. Vectors

Q47.

5(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π.
	$-3(-2)+2(3)+3(3)=21$	**M1**	Substitutes point on Π.
	$-3x+2y+3z=21$	**A1**	
		4	
5(b)	$\begin{pmatrix} 2 \\ -3 \\ 5+t \end{pmatrix}$	**B1**	Forms general point on line (given as a single vector).
	$-3(2)+2(-3)+3(5+t)=21$ leading to $t=6$	**M1**	Substitutes into the equation for Π.
	$\begin{pmatrix} 2 \\ -3 \\ 11 \end{pmatrix}$	**A1**	
		3	
5(c)	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \sqrt{1}\sqrt{22}\cos\alpha$ leading to $\cos\alpha = \frac{3}{\sqrt{22}}$	**M1** **A1 FT**	Uses dot product of **k** and their normal.
	Acute angle between l and Π is $90-\alpha=39.8^\circ$	**A1**	
		3	

5(d)	$\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$	**B1**	Finds direction vector from P to plane.
	$\frac{1}{\sqrt{22}}\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \frac{18}{\sqrt{22}} = 3.84$	**M1 A1**	Uses dot product of their direction and normal vectors.
		3	

Q48.

7(a)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\mathbf{r}.\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$	**A1**	
		3	
7(b)	$\frac{1}{\sqrt{1^2+3^2+2^2}} = \frac{1}{\sqrt{14}}$	**B1**	Divides by magnitude of the normal to Π 0.267
		1	
7(c)	$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = \sqrt{3}\sqrt{14}\cos\alpha$ leading to $\cos\alpha = \frac{6}{\sqrt{3}\sqrt{14}}$	**M1** **A1 FT**	Takes dot product of normal vectors.
	22.2°	**A1**	Accept 0.388 radians. Mark final answer.
		3	

7(d)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -7 \\ -2 & -3 & 1 \end{vmatrix} = \begin{pmatrix} -20 \\ 12 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**M1 A1**	Finds direction of common perpendicular.
	$\overrightarrow{OP} = \lambda \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 2-2\mu \\ 2-3\mu \\ \mu \end{pmatrix}$ leading to $\overrightarrow{PQ} = \begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix}$	**M1 A1**	Finds $\overrightarrow{PQ}$.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = 0$ or $\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} = k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**M1**	Uses that dot product of $\overrightarrow{PQ}$ with line direction is zero, or, alternatively, $\overrightarrow{PQ}$ is a multiple of the common perpendicular (parameter k not 1).
	$14\mu + 14\lambda = 10$	**A1**	Deduces one equation.
	$\begin{pmatrix} 2-2\mu-2\lambda \\ 2-3\mu-\lambda \\ \mu+7\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} = 0 \Rightarrow 14\mu + 54\lambda = 6$	**A1**	Deduces second equation.
	$\lambda = -\frac{1}{10}$ leading to $\overrightarrow{OP} = -\frac{1}{10}\begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix}$	**M1 A1**	Solves for λ and substitutes into $\overrightarrow{OP}$.
	$\mathbf{r} = \begin{pmatrix} -0.2 \\ -0.1 \\ 0.7 \end{pmatrix} + k \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$	**B1 FT**	FT using their common perpendicular.
		10	

Q49.

2(a)	$\overrightarrow{AB} = -8\mathbf{i} + 7\mathbf{j} - 5\mathbf{k} \quad \overrightarrow{AC} = 3\mathbf{j} - 3\mathbf{k}$	**B1**	Finds direction vectors of two lines in the plane.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix} = \begin{pmatrix} -6 \\ -24 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$	**M1 A1**	Finds normal to the plane *ABC*.
	$(4) + 4(-4) + 4(1) = -8 \Rightarrow x + 4y + 4z = -8$	**M1 A1**	Substitutes point.
	Alternative method for question 2(a)		
	Setting up 3 equations using the points given	**M1**	
	$x + 4y + 4z = -8$	**A1 A1 A1 A1**	
		5	
2(b)	$\frac{8}{\sqrt{1^2 + 4^2 + 4^2}} = 1.39$	**M1 A1**	Divides *their* constant by magnitude of *their* normal vector. $\frac{8}{\sqrt{33}}$ CAO
		2	
2(c)	$\mathbf{r} = t \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$	**B1**	Equation of line *OD*.
	$2t + 12t - 12t = -8$	**M1**	Substitutes into equation of plane.
	$(-8, -12, 12)$	**A1**	
		3	

6. Vectors

Q50.

7(a)	$\overrightarrow{AB} = 4\mathbf{i} - \mathbf{j} + \mathbf{k} \quad \overrightarrow{CD} = \mathbf{j} + (\lambda - 3)\mathbf{k}$	**B1**	Finds direction vectors of the two lines.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & \lambda - 3 \end{vmatrix} = \begin{pmatrix} 2 - \lambda \\ 12 - 4\lambda \\ 4 \end{pmatrix}$	**M1 A1**	Finds common perpendicular. This may also be done by setting up simultaneous equations and solving them.
	$\left\lvert \frac{1}{\sqrt{(\lambda-2)^2 + (4\lambda-12)^2 + 16}} \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 2 - \lambda \\ 12 - 4\lambda \\ 4 \end{pmatrix} \right\rvert$	**M1 A1**	Uses formula for perpendicular distance.
	$\left\lvert \frac{30 - 3\lambda}{\sqrt{17\lambda^2 - 100\lambda + 164}} \right\rvert = 3 \Rightarrow 9(\lambda - 10)^2 = 9\left(17\lambda^2 - 100\lambda + 164\right)$	**M1**	Sets equal to 3 and forms in quadratic in λ.
	$16\lambda^2 - 80\lambda + 64 = 0$ leading to $\lambda^2 - 5\lambda + 4 = 0$	**A1**	AG
		7	
7(b)(i)	$\mathbf{r} = 7\mathbf{i} + 4\mathbf{j} - \mathbf{k} + s(4\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(-5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$	**M1 A1**	OE M1 for using a correct point and attempting to find relevant direction vectors.
		2	
7(b)(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -5 & 3 & 5 \end{vmatrix} = \begin{pmatrix} -8 \\ -25 \\ 7 \end{pmatrix}$	**M1 A1**	Finds normal to the plane Π_2.
	$-8(7) - 25(4) + 7(-1)$ leading to $-8x - 25y + 7z = -163$	**M1 A1**	Substitutes point.
		4	

7(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -5 & 3 & 2 \end{vmatrix} = \begin{pmatrix} -5 \\ -13 \\ 7 \end{pmatrix}$	**M1 A1**	Finds normal to the plane Π_1.
	$\begin{pmatrix} -5 \\ -13 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} -8 \\ -25 \\ 7 \end{pmatrix} = \sqrt{243}\sqrt{738}\cos\theta$ leading to $\cos\theta = \frac{414}{\sqrt{243}\sqrt{738}}$	**M1 A1**	Uses dot product of normal vectors. $\cos = 0.9776176...$
	12.1°	**A1**	Mark final answer. Accept 0.212^c
		5	

6. Vectors

Q51.

4(a)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 2 \\ -8 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$	**M1 A1**	Finds vector perpendicular to Π.
	$2(3)-(-2)+4(1)=12$	**M1**	Substitutes point on Π.
	$2x-y+4z=12$	**A1**	
		4	
4(b)	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \sqrt{2}\sqrt{21}\cos\alpha$ leading to $\cos\alpha = \frac{3}{\sqrt{42}}$	**M1 A1 FT**	Uses dot product of $\mathbf{j}+\mathbf{k}$ and their normal. May be implied by angle 62.4°
	Acute angle between l and Π is $90-\alpha = 27.6^\circ$	**A1**	0.48 radians
		3	
4(c)	$\overrightarrow{OF} = \overrightarrow{OP} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+2t \\ 3-t \\ 1+4t \end{pmatrix}$	**M1 A1**	Forms $\overrightarrow{OF}$ using parameter.
	$21t+5=12$	**M1**	Substitutes into equation of plane.
	$\overrightarrow{OF} = \frac{1}{3}\begin{pmatrix} 8 \\ 8 \\ 7 \end{pmatrix}$	**A1**	
		4	

Q52.

6(a)	$\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix}$	**B1**	Finds a vector joining any point of l_1 to any point of l_2.
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 1 & -1 & 2 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}$	**M1 A1**	Finds common perpendicular.
	$\frac{1}{\sqrt{29}}\left\lvert \begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} \right\rvert = \frac{27}{\sqrt{29}} = 5.01$	**M1 A1**	Uses formula for shortest distance.
		5	
6(b)(i)	$\mathbf{r} = 2\mathbf{i}+\mathbf{k}+s(\mathbf{i}-\mathbf{j}+2\mathbf{k})+t(2\mathbf{i}-4\mathbf{j}-3\mathbf{k})$	**B1**	FT *their* common perpendicular from part **(a)**.
		1	
6(b)(ii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 2 & -4 & -3 \end{vmatrix} = \begin{pmatrix} -14 \\ -1 \\ -8 \end{pmatrix}$	**M1 A1FT**	Finds normal to the plane Π_2. FT *their* common perpendicular from part **(a)**.
	$14(0)+(2)+8(6)=50 \Rightarrow 14x+y+8z=50$	**M1 A1**	Substitutes point.
		4	

6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -4 & -3 \end{vmatrix} = \begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix}$	**M1 A1 FT**	Finds normal to the plane Π_1. FT *their* part **(b)(i).**
	$\begin{pmatrix} 11 \\ 7 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 14 \\ 1 \\ 8 \end{pmatrix} = \sqrt{174}\sqrt{261}\cos\theta$ leading to $\cos\theta = \dfrac{145}{\sqrt{174}\sqrt{261}}$	**M1 A1**	Uses dot product of normal vectors.
	47.1°	**A1**	0.822 radians
		5	

7. Proof by induction

Q1.

7 Prove by induction that

$$\sum_{r=1}^{n}(3r^5 + r^3) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2\mathrm{Q}(n),$$

where $\mathrm{Q}(n)$ is a quadratic function of n which is to be determined. [3]

9231/01/M/J/08 Q7

Q2.

9 Use induction to prove that

$$\sum_{n=1}^{N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(N+1)(2N+1)}.$$ [6]

Show that

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} < \frac{3}{8N^2}.$$ [4]

9231/01/O/N/08 Q9

Q3.

7 Let

$$I_n = \int_0^1 t^n \mathrm{e}^{-t}\,\mathrm{d}t,$$

where $n \geqslant 0$. Show that, for all $n \geqslant 1$,

$$I_n = nI_{n-1} - \mathrm{e}^{-1}.$$ [3]

Hence prove by induction that, for all positive integers n,

$$I_n < n!.$$ [5]

9231/01/M/J/09 Q7

7. Proof by induction

Q4.

Prove by induction that

$$\sum_{n=1}^{N} n^3 = \tfrac{1}{4}N^2(N+1)^2. \qquad [5]$$

Use this result, together with the formula for $\sum_{n=1}^{N} n^2$, to show that

$$\sum_{n=1}^{N}(20n^3 + 36n^2) = N(N+1)(N+3)(5N+2). \qquad [3]$$

Let

$$S_N = \sum_{n=1}^{N}(20n^3 + 36n^2 + \mu n).$$

Find the value of the constant μ such that S_N is of the form $N^2(N+1)(aN+b)$, where the constants a and b are to be determined. [3]

Show that, for this value of μ,

$$5 + \frac{22}{N} < N^{-4}S_N < 5 + \frac{23}{N},$$

for all $N \geqslant 18$. [3]

9231/01/O/N/09 Q11

Q5.

Let

$$I_n = \int_1^{\mathrm{e}} x(\ln x)^n \, \mathrm{d}x,$$

where $n \geqslant 1$. Show that

$$I_{n+1} = \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}(n+1)I_n. \qquad [3]$$

Hence prove by induction that, for all positive integers n, I_n is of the form $A_n\mathrm{e}^2 + B_n$, where A_n and B_n are rational numbers. [6]

9231/11/M/J/10 Q5

7. Proof by induction

Q6.

3 The sequence $x_1, x_2, x_3, \ldots$ is such that $x_1 = 3$ and

$$x_{n+1} = \frac{2x_n^2 + 4x_n - 2}{2x_n + 3}$$

for $n = 1, 2, 3, \ldots$. Prove by induction that $x_n > 2$ for all n. [6]

9231/13/M/J/10 Q3

Q7.

4 Prove by mathematical induction that, for all non-negative integers n, $7^{2n+1} + 5^{n+3}$ is divisible by 44. [5]

9231/01/O/N/10 Q4

Q8.

4 It is given that $f(n) = 3^{3n} + 6^{n-1}$.

(i) Show that $f(n+1) + f(n) = 28(3^{3n}) + 7(6^{n-1})$. [2]

(ii) Hence, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 7 for every positive integer n. [4]

9231/11/M/J/11 Q4

Q9.

2 Let $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Prove by mathematical induction that, for every positive integer n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}.$$ [5]

9231/13/M/J/11 Q2

Q10.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{1}{2}n} e^x \sin\left(x + \tfrac{1}{4}n\pi\right).$$ [7]

9231/11/O/N/11 Q3

Q11.

2 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!2^n}{(2x+3)^{n+1}}.$$ [6]

9231/13/O/N/11 Q2

7. Proof by induction

Q12.

2 Prove, by mathematical induction, that, for integers $n \geqslant 2$,

$$4^n > 2^n + 3^n.$$ [5]

9231/11/M/J/12 Q2

Q13.

2 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 1$ and $u_{r+1} = \frac{3u_r - 2}{4}$ for all r. Prove by mathematical induction that $u_n = 4\left(\frac{3}{4}\right)^n - 2$, for all positive integers n. [5]

9231/13/M/J/12 Q2

Q14.

5 Let I_n denote $\int_0^\infty x^n e^{-2x}\,dx$. Show that $I_n = \frac{1}{2}nI_{n-1}$, for $n \geqslant 1$. [2]

Prove by mathematical induction that, for all positive integers n, $I_n = \frac{n!}{2^{n+1}}$. [6]

9231/11/O/N/12 Q5

Q15.

3 Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers N,

$$S_N = 1 - \frac{1}{(N+1)!}.$$ [5]

9231/13/O/N/12 Q3

Q16.

2 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for every positive integer n. [5]

9231/11/M/J/13 Q2

Q17.

3 Prove by mathematical induction that, for every positive integer n,

$$\frac{d^n}{dx^n}(e^x \sin x) = (\sqrt{2})^n e^x \sin(x + \tfrac{1}{4}n\pi).$$ [7]

9231/13/M/J/13 Q3

7. Proof by induction

Q18.

5 It is given that $y = (1+x)^2 \ln(1+x)$. Find $\dfrac{d^3y}{dx^3}$. [3]

Prove by mathematical induction that, for every integer $n \geqslant 3$,

$$\frac{d^n y}{dx^n} = (-1)^{n-1} \frac{2(n-3)!}{(1+x)^{n-2}}.$$ [5]

9231/11/O/N/13 Q5

Q19.

9 Prove by mathematical induction that, for every positive integer n,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$ [5]

Express $\sin^5\theta$ in the form $p\sin 5\theta + q\sin 3\theta + r\sin\theta$, where p, q and r are rational numbers to be determined. [6]

9231/13/O/N/13 Q9

Q20.

3 Prove by mathematical induction that, for all non-negative integers n,

$$11^{2n} + 25^n + 22$$

is divisible by 24. [6]

9231/11/M/J/14 Q3

Q21.

3 It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3, \ldots$. Prove, by mathematical induction, that $\phi(n)$ is divisible by 8, for every positive integer n. [7]

9231/13/M/J/14 Q3

Q22.

3 It is given that $u_r = r \times r!$ for $r = 1, 2, 3, \ldots$. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$. Write down the values of

$$2! - S_1, \quad 3! - S_2, \quad 4! - S_3, \quad 5! - S_4.$$ [2]

Conjecture a formula for S_n. [1]

Prove, by mathematical induction, a formula for S_n, for all positive integers n. [4]

9231/11/O/N/14 Q3

7. Proof by induction

Q23.

3 The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n.

Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

9231/11/M/J/15 Q3

Q24.

3 Prove by mathematical induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$. [6]

State the value of $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

9231/13/M/J/15 Q3

Q25.

3 Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}.$$ [6]

9231/11/O/N/15 Q3

Q26.

3 Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

9231/11/M/J/16 Q3

Q27.

2 It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geqslant 3$. [6]

9231/13/M/J/16 Q2

Q28.

4 Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

9231/11/O/N/16 Q4

7. Proof by induction

Q29.

2 Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

9231/11/M/J/17 Q2

Q30.

3 Prove, by mathematical induction, that $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

9231/13/M/J/17 Q3

Q31.

3 **(i)** Show that $\frac{d^{n+1}}{dx^{n+1}}(x^{n+1} \ln x) = \frac{d^n}{dx^n}(x^n + (n+1)x^n \ln x)$. [2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$ [5]

9231/11/O/N/17 Q3

Q32.

2 It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

9231/11/M/J/18 Q2

7. Proof by induction

Q33.

9 For the sequence $u_1, u_2, u_3, \ldots$, it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \ldots + (u_r - 3)x^r + \ldots$$

is convergent. [2]

(iii) Use the result given in part **(i)** to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$ [3]

9231/13/M/J/18 Q9

Q34.

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

9231/11/O/N/18 Q3

Q35.

6 It is given that $y = e^x u$, where u is a function of x. The rth derivatives $\frac{d^r y}{dx^r}$ and $\frac{d^r u}{dx^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = e^x\left(\binom{n}{0}u + \binom{n}{1}u^{(1)} + \binom{n}{2}u^{(2)} + \ldots + \binom{n}{r}u^{(r)} + \ldots + \binom{n}{n}u^{(n)}\right).$$ [8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

9231/12/O/N/18 Q6

7. Proof by induction

Q36.

8 **(i)** Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$ [5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} (\tfrac{1}{2})^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$ [5]

9231/11/M/J/19 Q8

Q37.

1 Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n. [5]

9231/13/M/J/19 Q1

Q38.

2 It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{(n-1)!a^n}{(ax+1)^n}.$$ [6]

9231/11/O/N/19 Q2

Q39.

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

9231/13/M/J/20 Q2

Q40.

5 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right).$$ [7]

9231/11/O/N/20 Q5

Q41.

2 Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

9231/12/O/N/20 Q2

7. Proof by induction

Q42.

1 Prove by mathematical induction that $2^{4n}+31^n-2$ is divisible by 15 for all positive integers n. [6]

9231/11/M/J/21 Q1

Q43.

3 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

9231/13/M/J/21 Q3

Q44.

The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1}=\left(a_n+\frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1}\ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x+\frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1}\ln 4$ for $n \geqslant 2$. [2]

9231/11/O/N/21 Q3

Q45.

It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n}=\left(a^n x+na^{n-1}\right)e^{ax}.$$ [6]

9231/12/O/N/21 Q2

7. Proof by induction

Q46.

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

9231/11/M/J/22 Q3

Q47.

Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

9231/13/M/J/22 Q5

Q48.

Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

9231/11/O/N/22 Q2

Q49.

The function f is such that $\mathrm{f}''(x) = \mathrm{f}(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}\left(x\mathrm{f}(x)\right) = x\mathrm{f}'(x) + (2n-1)\mathrm{f}(x).$$ [7]

9231/12/O/N/22 Q4

7. Proof by induction

Solutions

Q1.

7 Verifies H_1 to be true B1

$H_k : \sum_{r=1}^{k} (3r^5 + r^3) = (1/2)k^3(k+1)^3$ B1

$H_k \Rightarrow \sum_{r=1}^{k+1} (3r^5 + r^3) = (1/2)k^3(k+1)^3 + 3(k+1)^5 + (k+1)^3$ M1

$= \ldots = (1/2)(k+1)^3(k+2)^3$ A1

Thus $H_k \Rightarrow H_{k+1}$ and concludes A1

$3\sum_{r=1}^{n} r^5 + (1/4)n^2(n+1)^2 = (1/2)n^3(n+1)^3$ M1

$\Rightarrow \ldots \Rightarrow \sum_{r=1}^{n} r^5 = (1/12)n^2(n+1)^2(2n^2+2n-1)$ M1A1

Q2.

9 Set up

$$H_k : \sum_{n=1}^{k} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)}$$ B1

for some positive integer k

$$H_k \Rightarrow \sum_{n=1}^{k+1} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = 1 - \frac{1}{(k+1)(2k+1)} + \frac{4k+5}{(k+1)(k+2)(2k+1)(2k+3)}$$ M1

$$= 1 - \frac{2k^2+3k+1}{(k+1)(k+2)(2k+1)(2k+3)}$$ A1

$$= \ldots = 1 - \frac{1}{(k+2)(2k+3)}$$ A1

Verifies H_1 is true. B1

Correct completion of induction argument A1

$$\sum_{n=N+1}^{2N} \frac{4n+1}{n(n+1)(2n-1)(2n+1)} = \ldots = \frac{1}{(N+1)(2N+1)} - \frac{1}{(2N+1)(4N+1)}$$ M1A1

$$= \frac{3N}{(N+1)(2N+1)(4N+1)} < \frac{3N}{N.2N.4N} = \frac{3}{8N^2}$$ M1A1

OR

$$= \frac{3N}{8N^3 + 14N^2 + 7N + 1} = \frac{3}{8N^2 + 14N + 7 + \frac{1}{N}}$$

Since $N \geqslant 1 \quad 14N + 7 + \frac{1}{N} > 0$

$$\therefore \sum < \frac{3}{8N^2}$$

Q3.

7	$I_n = \left[-t^n e^{-t}\right]_0^1 + n\int_0^1 t^{n-1}e^{-t}\,dt$	M1A1
	$I_n = nI_{n-1} - e^{-1}$ (AG)	A1
	$H_k : I_k < k!$ for some k	B1
	$H_k : \Rightarrow I_{k+1} < (k+1)k! - e^{-1} < (k+1)!$	M1A1
	$I_1 = 1 - 2e^{-1}$	B1
	(Do **not** allow I_0 unless I_1 is deduced from it)	
	Completes induction argument	A1
	This requires at least:	
	'The assumption is true and it is true for all positive integers' for the final mark, which is only awarded if all previous marks have been gained in the proof part.	

7. Proof by induction

Q4.

11 EITHER

$H_k : S_k = \sum_{n=1}^{k} n^3 = (1/4)k^2(k+1)^2$ for some k — B1

$H_k \Rightarrow S_{k+1} = (1/4)k^2(k+1)^2 + (k+1)^3$ — M1

$= \ldots = (1/4)(k+1)^2(k+2)^2$ so that $H_k \Rightarrow H_{k+1}$ — M1A1

Verifies H_1 is true and completes induction argument — A1

$\sum_{n=1}^{N}(20n^3 + 36n^2) = 5N^2(N+1)^2 + 6N(N+1)(2N+1)$ — M1

$= \ldots = N(N+1)(N+3)(5N+2)$ (AG) — M1A1

$S_N = N(N+1)(N+3)(5N+2) + (\mu/2)N(N+1)$ — M1

$= N(N+1)(5N^2 + 17N + 6 + \mu/2)$ — M1

Take $\mu = -12$, then $S_N = N^2(N+1)(5N+17)$ so that $a = 5$, $b = 17$ — A1

$N^{-4}S_N = 5 + 22/N + 17/N^2$, $> 5 + 22/N$, $\forall N \geqslant 1$ — M1, A1

$N \geqslant 18 \Rightarrow N > 17 \Rightarrow 17/N^2 < 1/N$

$\Rightarrow N^{-4}S_N < 5 + 23/N$ (AG) — A1

Q5.

5 $D[(x^2/2)(\ln x)^n] = x(\ln x)^n + (nx/2)(\ln x)^{n-1}$ — M1

$\Rightarrow [(x^2/2)(\ln x)^n]_1^e = I_n + (n/2)I_{n-1}$ — A1

$\Rightarrow \ldots \Rightarrow I_n = e^2/2 - (n/2)I_{n-1}$ for $n \geqslant 2$

$\Rightarrow I_{n+1} = e^2/2 - (n+1)2I_{n-1}$ for $n \geqslant 1$ — A1

[3]

OR

$\int_1^e x(\ln x)^n\,dx = [(x^2/2)(\ln x)^n]_1^e - \int_1^e (nx/2)(\ln x)^{n-1}\,dx$ — M1A1

$\Rightarrow I_n = e^2/2 - (n/2)I_{n-1} \quad \Rightarrow I_{n+1} = \frac{e^2}{2} - \frac{(n+1)}{2}I_n$ — A1

$H_k : I_k = A_k e^2 + B_k$, where A_k and B_k are rational — B1

$H_k \Rightarrow I_{k+1} = e^2/2 - (k+1)(A_k e^2 + B_k)/2$,

$= A_{k+1}e^2 + B_{k+1}$, where $A_{k+1} = 1/2 - (k+1)A_k/2$, $B_{k+1} = -(k+1)B_k/2$, — M1

$\Rightarrow A_{k+1}$ and B_{k+1} are rational — A1

$I_1 = e^2/4 + 1/4 \Rightarrow A_1 = 1/4$, $B_1 = 1/4 \Rightarrow H_1$ is true — M1A1

Completion of induction argument — A1

[6]

7. Proof by induction

Q6.

3	$H_k : x_k > 2$ for some k	B1
	$x_{k+1} - 2 = \left(2x_k^2 - 8\right)/\left(2x_k + 3\right)$	M1A1
	$H_k \Rightarrow 2x_k^2 - 8 > 0 \Rightarrow x_{k+1} > 2 \Rightarrow H_{k+1}$	A1
	$x_1 = 3 > 2 \Rightarrow H_1$ is true	B1 CWO
	Completion of the induction argument	A1 [6]

Alternatively for lines 2 and 3:

$x_{k+1} = x_k + \frac{1}{2} - \frac{3\frac{1}{2}}{\left(2x_k + 3\right)}$	M1A1
$H_k \Rightarrow 2x_k + 3 > 7 \Rightarrow H_{k+1}$	A1
OR $x_{k+1} = x_k + \frac{x_{k-2}}{\left(2x_k + 3\right)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > 2$	A1
OR $x_{k+1} - x_k = \frac{x_{k-2}}{\left(2x_k + 3\right)}$	M1A1
$x_k > 2 \Rightarrow x_{k+1} > x_k > 2$	A1
Minimum conclusion is 'Hence true for $n \geqslant 1$'.	

Q7.

4	$n = 0$: $7^1 + 5^3 = 132$ which is divisible by 44	B1		
	Assume $7^{2k+1} + 5^{k+3}$ is divisible by 44	B1		
	Consider $7^{2(k+1)+1} + 5^{(k+1)+3} = 7^2 7^{2k+1} + 5.5^{k+3}$	M1	$(k+1)$ th term	
	$= 49(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	[5]
	Alternative solution for final three marks:			
	Consider $(7^{2k+3} + 5^{k+4}) - (7^{2k+1} + 5^{k+3})$	M1		
	$= 48(7^{2k+1} + 5^{k+3}) - 44.5^{k+3}$	M1	in appropriate form	
	which is divisible by 44	A1	convincing argument	

7. Proof by induction

Q8.

4 (i)	Establishes initial result.	$f(n)+f(n+1)=3^{3n}+6^{n-1}+3^{3n+3}+6^{n}$ $=3^{3n}(1+27)+6^{n-1}(1+6)$ $=28(3^{3n})+7(6^{n-1})$ (AG)	M1 A1	2	
(ii)	States inductive hypothesis.	H_k: $f(k)=7\lambda$	B1		
	Proves base case.	$3^3+6^0=28=4\times7\Rightarrow H_1$ is true	B1		
	Shows $P_k \Rightarrow P_{k+1}$.	$f(k+1)+f(k)=f(k+1)+7\lambda=28(3^{3k})+7(6^{k-1})$ $=7\mu$ $\Rightarrow f(k+1)=7(\mu-\lambda)\ \therefore H_k \Rightarrow H_{k+1}$	M1		
	States conclusion.	(Hence by the principle of mathematical induction H_n is) **true for all positive integers** n.	A1	4	**[6]**

Q9.

2	States proposition.	Let P_n be the proposition: $\mathbf{A}=\begin{pmatrix}2&3\\0&1\end{pmatrix}\Rightarrow\mathbf{A}^n=\begin{pmatrix}2^n&3(2^n-1)\\0&1\end{pmatrix}$			
	Shows base case is true.	$\mathbf{A}^1=\begin{pmatrix}2&3\\0&1\end{pmatrix}=\begin{pmatrix}2^1&3\times(2-1)\\0&1\end{pmatrix}\Rightarrow P_1$ is true.	B1		
		Assume P_k is true for some integer k.	B1		
	Proves inductive step.	$\mathbf{A}^{k+1}=\begin{pmatrix}2&3\\0&1\end{pmatrix}\begin{pmatrix}2^k&3(2^k-1)\\0&1\end{pmatrix}$ $=\begin{pmatrix}2^{k+1}&3.2(2^k-1)+3\\0&1\end{pmatrix}$ $=\begin{pmatrix}2^{k+1}&3(2^{k+1}-1)\\0&1\end{pmatrix}$	M1 A1		
	States conclusion.	Since P_1 is true and $P_k \Rightarrow P_{k+1}$, hence by PMI P_n is true $\forall$ positive integers n.	A1	5	**[5]**

7. Proof by induction

Q10.

3	Proves base case.	$H_n: \frac{d^n}{dx^n}(e^x \sin x) = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$			
		$\frac{d}{dx}(e^x \sin x) = \sin x e^x + e^x \cos x$	M1		
		$= \sqrt{2} e^x \left(\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}\right) = 2^{\frac{1}{2}} e^x \sin\left(x + \frac{\pi}{4}\right)$			
		$\Rightarrow H_1$ is true.	A1		
	States inductive hypothesis.	Assume H_k is true :	B1		
	Proves inductive step.	$\frac{d^{k+1}}{dx^{k+1}}(e^x \sin x) = 2^{\frac{k}{2}} \left\{e^x \sin\left(x + \frac{k\pi}{4}\right) + e^x \cos\left(x + \frac{k\pi}{4}\right)\right\}$	M1		
		$= 2^{\frac{k+1}{2}} e^x \left\{\frac{1}{\sqrt{2}} \sin\left(x + \frac{k\pi}{4}\right) + \frac{1}{\sqrt{2}} \cos\left(x + \frac{k\pi}{4}\right)\right\}$	A1		
		$= 2^{\frac{k+1}{2}} e^x \left\{\sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)\right\}$			
		$= 2^{\frac{k+1}{2}} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$	A1		
	States conclusion.	$\therefore H_k \Rightarrow H_{k+1}$ Hence true for all positive integers by PMI	A1	7	**[7]**

Q11.

2	States proposition.	$P_n: \frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n! 2^n}{(2x+3)^{n+1}}$			
	Proves base case.	$\frac{d}{dx}\left(\frac{1}{2x+3}\right) = (-1)(2x+3)^{-2} \times 2$	M1		
		$= (-1)\frac{1! \times 2}{(2x+3)^2} \Rightarrow P_1$ is true.	A1		
	States inductive hypothesis.	Assume P_k is true.			
		i.e. $\frac{d^k}{dx^k}\left(\frac{1}{2x+3}\right) = (-1)^k \frac{k! 2^k}{(2x+3)^{k+1}}$	B1		
	Shows $P_k \Rightarrow P_{k+1}$.	$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{2x+3}\right) = (-1)^{k+1} \frac{2(k+1) k! 2^k}{(2x+3)^{k+2}}$	M1		
		$= (-1)^{k+1} \frac{(k+1)! 2^{k+1}}{(2x+3)^{k+2}}$	A1		
		$\therefore P_k \Rightarrow P_{k+1}$			
	States conclusion.	Since P_1 is true and $P_k \Rightarrow P_{k+1}$, hence by the principle of mathematical induction P_n is true $\forall n \in Z^+$.	A1	6	**[6]**

7. Proof by induction

Q12.

2	(States proposition.)	$(P_n: 4^n > 2^n + 3^n)$			
	Proves base case.	Let $n = 2$, $16 > 4 + 9 \Rightarrow P_2$ is true.	B1		
	States inductive hypothesis.	Assume P_k is true $\Rightarrow 4^k > 2^k + 3^k$	B1		
	Proves inductive step.	$4^{k+1} = 4.4^k > 4(2^k + 3^k) = 4.2^k + 4.3^k$	M1		
		$> 2.2^k + 3.3^k = 2^{k+1} + 3^{k+1}$	A1		
		$\therefore P_k \Rightarrow P_{k+1}$			
	States conclusion.	Hence result true, by PMI, for all integers $n \geq 2$.	A1 (CWO)	5	**[5]**

Q13.

2	(States proposition.)	$(P_n: u_n = 4\left(\frac{3}{4}\right)^n - 2)$			
	Proves base case.	Let $n = 1$ $4 \times \frac{3}{4} - 2 = 3 - 2 = 1 \Rightarrow P_1$ true.	B1		
	States Inductive hypothesis.	Assume P_k is true for some k.	B1		
	Proves inductive step.	$u_{k+1} = \frac{3\left\{4\left(\frac{3}{4}\right)^k - 2\right\} - 2}{4} = 4.\frac{3}{4}.\left(\frac{3}{4}\right)^k - \frac{6+2}{4}$	M1		
		$= 4.\left(\frac{3}{4}\right)^{k+1} - 2 \quad \therefore P_k \Rightarrow P_{k+1}$	A1		
	States conclusion.	$\therefore$ By PMI P_n is true $\forall$ positive integers.	A1	5	**[5]**

7. Proof by induction

Q14.

5	Integrates by parts to obtain reduction formula.	$\int_0^\infty x^n e^{-2x}dx = \left[x^n \frac{-e^{-2x}}{2}\right]_0^\infty + \int_0^\infty nx^{n-1}\frac{e^{-2x}}{2}dx$	M1		
		$= \frac{n}{2}I_{n-1}$ (AG)	A1	2	
	(States proposition.)	$P_n: I_n = \frac{n!}{2^{n+1}}$			
	Proves base case.	$n=1 \quad I_0 = \int_0^\infty e^{-2x}dx = \left[-\frac{1}{2}e^{-2x}\right]_0^\infty = \frac{1}{2}$	B1		
		$I_1 = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4} = \frac{1!}{2^2}$ $\therefore P_1$ true.	B1		
	Shows $P_k \Rightarrow P_{k+1}$.	$P_k: I_k = \frac{k!}{2^{k+1}}$ for some integer k.	B1		
		$\therefore I_{k+1} = \frac{k+1}{2}\times\frac{k!}{2^{k+1}} = \frac{(k+1)!}{2^{k+2}}$	M1A1		
		$\therefore P_k \Rightarrow P_{k+1}$			
	States conclusion.	Hence by PMI P_n is true for all positive integers n.	A1	6	**[8]**

Q15.

3	Proposition.	$H_N: S_N = 1 - \frac{1}{(N+1)!}$			
	Proves base case.	$S_1 = \frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!} \Rightarrow H_1$ is true.	B1		
	States inductive hypothesis.	H_k : Assume $S_k = 1 - \frac{1}{(k+1)!}$ is true.	B1		
	Proves inductive step.	$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)!-(k+2)+(k+1)}{(k+2)!}$	M1		
		$\Rightarrow S_{k+1} = 1 - \frac{1}{(k+2)!}$ $\therefore H_k \Rightarrow H_{k+1}$.	A1		
	States conclusion.	$\therefore$ (By PMI H_n is) true for all positive integers N.	A1	5	**[5]**

Q16.

2	Proves base case.	P_n: $5^{2n}-1$ is divisible by 8.	B1		
		$5^2 - 1 = 24 = 3\times 8 \Rightarrow P_1$ is true	B1		
	States inductive hypothesis.	Assume P_k is true: $5^{2k}-1 = 8\lambda$ for some k.	M1		
		$5^{2k+2}-1 = 25.5^{2k}-1 = 24.5^{2k}+5^{2k}-1$			
		$= 3\times 8.5^{2k} + 8\lambda$			
	Proves inductive step.	$\therefore P_k \Rightarrow P_{k+1}$	A1		
	States conclusion.	(Since P_1 is true and $P_k \Rightarrow P_{k+1}$) $\therefore$ P_n is **for every positive integer** n (by PMI).	A1	5	**[5]**

7. Proof by induction

Q17.

3	Differentiates once.	$\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x$	B1		
	Rearranges.	$= \sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$	M1		
	Shows true for $n = 1$.	$= \sqrt{2}e^x \sin\left(x + \frac{1}{4}\pi\right) \Rightarrow H_1$ true.	A1		
	States inductive hypothesis. (May be seen by implication.)	$H_k : \frac{d^k}{dx^k}(e^x \sin x) = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{1}{4}k\pi\right)$	B1		
	Differentiates.	$\frac{d^{k+1}}{dx^{k+1}} = \left(\sqrt{2}\right)^k\left(\sin\left(x + \frac{1}{4}k\pi\right)e^x + e^x \cos\left(x + \frac{1}{4}k\pi\right)\right)$	M1		
	Rearranges.	$= \left(\sqrt{2}\right)^{k+1} e^x\left(\frac{1}{\sqrt{2}}\sin\left(x + \frac{1}{4}k\pi\right) + \frac{1}{\sqrt{2}}\cos\left(x + \frac{1}{4}k\pi\right)\right)$	A1	7	**[7]**

	Shows $H_k \Rightarrow H_{k+1}$	$= \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + \frac{1}{4}(k+1)\pi\right) \Rightarrow H_{k+1}$ true.	A1		
	and states conclusion.	$\therefore$, by PMI, true for all positive integers n. (CWO)			

Q18.

5 (i)	Differentiates once,	$y' = 2(1+x)\ln(1+x) + (1+x)$	B1		
	twice	$y'' = 2\ln(1+x) + 3$	B1		
	and three times.	$y''' = \frac{2}{1+x}$ (Allow B1√ if constant term in previous line incorrect.)	B1	3	
(ii)	Proves base case.	$\frac{d^3y}{dx^3} = \frac{(-1)^2.2.0!}{1+x} = \frac{2}{1+x} \Rightarrow H_3$ is true.	B1		
	States inductive hypothesis.	$H_k : \frac{d^k y}{dx^k} = \frac{(-1)^{k-1}.2.(k-3)!}{(1+x)^{k-2}}$ for some k.	B1		
	Differentiates	$\frac{d^{k+1}y}{dx^{k+1}} = (-1)^{k-1}.2(k-3)!(-1)(k-2)(1+x)^{-(k-1)}$	M1		
	Proves inductive step and states conclusion.	$= \frac{(-1)^k.2.(k-2)!}{(1+x)^{k-1}} \Rightarrow H_{k+1}$ is true	A1		
		Hence by PMI H_n is true for all integers $\geqslant 3$	A1	5	**[8]**

7. Proof by induction

Q19.

9	States inductive hypothesis	H_k: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ for some k	B1		
	Proves $H_k \Rightarrow H_{k+1}$	$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$ $= (\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)$ $= \cos[k+1]\theta + i\sin[k+1]\theta \quad \therefore H_k \Rightarrow H_{k+1}$	M1 A1 A1		
	States conclusion	H_1 is trivially true, so true for all positive integers	A1	**(5)**	
	Uses de M's Thm. to find $2i\sin n\theta$	$z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$ $= 2i\sin n\theta$	B1		
	Uses binominal expansion	$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32i\sin^5\theta$	B1		
	and groups	$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$ $= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	M1 A1		
	Applies above result and obtains $\sin^5$	$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$ $\sin^5\theta = \frac{1}{16}(\sin^5\theta - 5\sin 3\theta + 10\sin\theta)$	M1 A1	**(6)**	**[11]**

Q20.

3	H_k: $f(k) = 11^{2k} + 25^k + 22 = 24\lambda$	B1	
	$f(0) = 1 + 1 + 22 = 24 = 1 \times 24 \Rightarrow H_0$ is true.	B1	
	$f(k+1) - f(k) = 11^{2k+2} + 25^{k+1} + 22 - (11^{2k} + 25^k + 22)$ $= 11^{2k}(121 - 1) + 25^k(25 - 1)$ $= 11^{2k} \times 24 \times 5 + 25^k \times 24 = 24\mu$	M1 A1 A1	
	Alternatively:		
	$f(k+1) = 11^{2k+2} + 25^{k+1} + 22$ $= 121.11^{2k} + 25.25^k + 22 = (120 + 1)11^{2k} + (24 + 1)25^k + 22$ (OE) $= 120.11^{2k} + 24.25^k + 24\lambda = 24\mu$	(M1) (A1) (A1)	
	$\Rightarrow f(k+1) = 24\mu + 24\lambda = 24(\mu + \lambda) \Rightarrow H_{k+1}$ is true. Hence by PMI H_n is true for all non-negative integers. (Must see non-negative integers.) CSO: Final mark requires all previous marks.	A1 [6]	

7. Proof by induction

Q21.

3	$\phi(1) = 5 \times 5 - 1 = 24$ which is divisible by $8 \Rightarrow H_1$ is true.	B1
	Assume P_k is true for some positive integer $k \Rightarrow \phi(k) = 8l$	B1
	$\phi(k+1) - \phi(k) = 5^{k+1}(4k+5) - 1 - 5^k(4k+1) + 1$	M1
	$= 5^k(20k + 25 - 4k - 1)$	A1
	$= 5^k(16k + 24) = 8m$	A1
	$\therefore \phi(k+1) = 8(l+m)$	A1
	Hence, by PMI, true for all positive integers n. (CWO – all previous marks required.)	A1
		[7]
	Alternatively	
	$\phi(k+1) = 5^{k+1}(4k+5) - 1$	
	$= 5.\ (4k.5^k) + 25.5^k - 1$	
	$= 5(8l - 5^k + 1) + 25.5^k - 1$	(M1A1)
	$= 40l + 20.5^k + 4$	
	$= 40l + 24.5^k - 4.5^k + 4$	(A1)
	$= 40l + 24.5^k - 4(5^k - 1)$	
	$= 40l + 24.5^k - 4(8l - 4k.5^k)$	
	$= 8l + 24.5^k + 16k.5^k$	(A1)
	$= 8m$	

Q22.

3	$2! - S_1 = 1$, $3! - S_2 = 1$, $4! - S_3 = 1$, $5! - S_4 = 1$ (Two correct B1, all four correct B2)	B2,1,0
		(2)
	$S_n = (n+1)! - 1$	B1
		(1)
	$2! - 1 = 2 - 1 = 1 \Rightarrow H_1$ is true.	B1
	H_k: $S_k = (k+1)! - 1$	B1
	$(k+1)! - 1 + (k+1) \times (k+1)!$	
	$= (k+1)!(1 + k + 1) - 1$	M1
	$= ([k+1] + 1)! - 1$ Hence $H_k \Rightarrow H_{k+1}$	
	So result holds for all positive integers (by PMI).	A1
		(4)
		[7]

7. Proof by induction

Q23.

3	$a_1 > 5$ (given) $\Rightarrow H_1$ is true. Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$. $a_{k+1} - 5 = \frac{4a_k^2 + 25}{5a_k} - 5 = \frac{4a_k^2 + 25 - 25a_k}{5a_k} = \frac{(4a_k - 5)(a_k - 5)}{5a_k} > 0 \ , \Rightarrow a_{k+1} > 5$ **Or** $a_{k+1} = \frac{4}{5}(5+\delta) + \frac{5}{5+\delta}, \ = 4 + \frac{4}{5}\delta + (1 - \frac{\delta}{5} + \frac{\delta^2}{25} - ...)$ for $0 < \delta < 5$ $= 5 + \frac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial). $H_k \Rightarrow H_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbf{Z}^+$ (N.B. The minimum requirement is 'true for all positive integers'.) $a_{k+1} - a_k = \frac{5}{a_k} - \frac{1}{5}a_k$ $\frac{5}{a_k} < 1$ and $\frac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	B1 B1 M1A1 (M1) (A1) A1 (5) M1 A1 (2) **Total: 7**

Q24.

3	$H_k : \sum_{r=1}^{k} \frac{1}{(2r)^2 - 1} = \frac{k}{2k+1}$ is true for some integer k. $\frac{1}{2^2 - 1} = \frac{1}{3} = \frac{1}{2 \times 1 + 1} \Rightarrow H_1$ is true. $\frac{k}{2k+1} + \frac{1}{(2k+2)^2 - 1} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$ $= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2[k+1]+1}$ $\therefore H_k \Rightarrow H_{k+1}$ $\therefore$ (By Principle of Mathematical Induction) H_n is true for all positive integers n. (This mark requires all previous marks.) $\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} = \frac{1}{2}$	B1 B1 M1A1 A1 A1 (6) B1 (1) **Total 7**

7. Proof by induction

Q25.

3	$n=1$ in formula gives $a^0e^{ax}+axe^{ax}=e^{ax}+axe^{ax}$	**B1**
	$\frac{d}{dx}\left(xe^{ax}\right)=e^{ax}\times1+x.ae^{ax}=e^{ax}+axe^{ax}\Rightarrow H_1$ is true oe	**B1**
	Assume H_k is true, i.e. $\frac{d^k}{dx^k}\left(xe^{ax}\right)=ka^{k-1}e^{ax}+a^kxe^{ax}$.	**B1**
	$\frac{d^{k+1}}{dx^{k+1}}\left(xe^{ax}\right)=ka^ke^{ax}+a^ke^{ax}+a^{k+1}xe^{ax}$	**M1**
	$=(k+1)a^ke^{ax}+a^{k+1}xe^{ax}$	**A1**
	$\Rightarrow H_{k+1}$ is true, hence by PMI H_n is true for all positive integers n.	**A1** [6] **Total 6**

Q26.

3	For $n=1\quad 10+192+5=207=9\times23\Rightarrow H_1$ is true.	**B1**
	Assume H_k is true for some positive integer $k\Rightarrow 10^n+3.4^{n+2}+5=9\alpha$	**B1**
	Let $f(n)=10^n+3.4^{n+2}+5$	
	Hence $f(n+1)-f(n)=10^n(10-1)+3.4^{n+2}(4-1)$	**M1**
	$=9\left(10^n+4^{n+2}\right)$	
	$=9\beta$	**A1**
	Hence $f(n+1)\left(=9(\beta+\alpha)\right)\Rightarrow H_{k+1}$ is true	**A1**
	H_1 is true and $H_k\Rightarrow H_{k+1}$, hence by PMI H_n is true for all positive integers n.	**A1**
	N.B. Or can show $f(n+1)=9\left(10\alpha-2.4^{n+2}-5\right)$ for **M1A1A1**. (3rd,4th&5th marks)	[6]

Q27.

2	With $n=3$, $\frac{1}{2}n(n-3)=0$	**M1**
	A triangle has no diagonals $\Rightarrow H_3$ is true.	**A1**
	Assume H_k is true: A k-gon has $\frac{1}{2}k(k-3)$ diagonals for some integer ≥ 3	**B1**
	Adding an extra vertex, a further $(k-1)$ diagonals can be drawn.	**M1**
	$\frac{1}{2}k(k-3)+k-1=\frac{k^2-3k+2k-2}{2}=\frac{(k+1)(k-2)}{2}$	**A1**
	$=\frac{1}{2}(k+1)(k+1-3)\quad$ (So $H_k\Rightarrow H_{k+1}$)	**A1**
	$\Rightarrow H_n$ is true for all integers $n\geqslant 3$.	[6]

7. Proof by induction

Q28.

4	$\binom{n}{r-1}+\binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{r!(n-r)!}=\frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1}+\frac{1}{r}\right)$ $=\frac{n!}{(r-1)!(n-r)!}\left(\frac{r+n-r+1}{r(n-r+1)}\right)=\frac{(n+1)!}{r!(n-r+1)!}=\binom{n+1}{r}$	**M1** **A1**	[2]
	$(a+x)^1=\binom{1}{0}a+\binom{1}{1}x=a+x \Rightarrow H_1$ is true.	**B1**	
	Assume H_k is true,i.e. $(a+x)^k=\binom{k}{0}a^k+\binom{k}{1}a^{k-1}x+\ldots+\binom{k}{r}a^{k-r}x^r+\ldots+\binom{k}{k}x^k$	**B1**	
	Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1}+\binom{k}{r}=\binom{k+1}{r}$ $\Rightarrow H_{k+1}$ is true.	**M1**	
	Hence H_n is true for all positive integers.	**A1**	[4]

Q29.

2	Let P_n be the proposition that 5^n+3 is divisible by 4 $5^0+3=4 \Rightarrow P_0$ is true (allow P_1)	**B1**	Some explanation of what P_k being true means
	Assume that P_k is true for some non-negative integer k.	**B1**	or e.g. $5^k+3=4\alpha$ for 2nd **B1**
	$5^{k+1}+3=5(4\alpha-3)+3$	**M1**	*Alt method:* Use f(k+1) – f(k) **M1 A1**
	$=20\alpha-12=4(5\alpha-3)$ (or shows that $5^{k+1}+3=5.5^k+5.3-4.3=5(5^k+3)-4.3$)	**A1**	
	P_0 is true and $P_k \Rightarrow P_{k+1}$, hence P_n is true for all non-negative integers n.	**A1**	
	Total:	**5**	

Q30.

3	When $n=1$ $\quad 1\times\ln 2=\ln 2=\ln\left(\frac{2^1}{1!}\right)\Rightarrow (H_1$ is true$)$	**B1**	
	Assume, for some positive integer k, that $\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{[k+1]^k}{k!}\right)$	**B1**	
	Hence $\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right)=\ln\left(\frac{(k+1)^k}{k!}\right)+(k+1)\ln\left(\frac{k+2}{k+1}\right)$	**B1**	
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{k!(k+1)^{k+1}}$	**M1**	
	$=\ln\frac{(k+1)^k(k+2)^{k+1}}{(k+1)!(k+1)^k}$ $=\ln\frac{(k+2)^{k+1}}{(k+1)!}$	**A1**	
	Thus $H_k \Rightarrow H_{k+1}$ and hence by PMI H_n is true for all positive integers.	**A1**	
	Total:	**6**	

7. Proof by induction

Q31.

3(i)	$\frac{d^{n+1}}{dx^{n+1}}\left(x^{n+1}\ln x\right)=\frac{d^n}{dx^n}\left(x^{n+1}.\frac{1}{x}+(n+1)x^n\ln x\right)=$ $\frac{d^n}{dx^n}\left(x^n+(n+1)x^n\ln x\right)$	**M1A1**	AG
		2	
3(ii)	Assume H_k is true $\Rightarrow \frac{d^k}{dx^k}\left(x^k\ln x\right)=k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$	**B1**	Statement of H_k seen
	$\frac{d^{k+1}}{dx^{k+1}}\left(x^{k+1}\ln x\right)=\frac{d^k}{dx^k}\left(x^k+[k+1]x^k\ln x\right)$	**M1**	
	$=k!+[k+1]k!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k}\right\}$	**A1**	
	$=(k+1)!\left\{\ln x+1+\frac{1}{2}+\ldots+\frac{1}{k+1}\right\}\Rightarrow H_{k+1}$ is true	**A1**	
	Check H_1 is true **and** H_k is true $\Rightarrow H_{k+1}$ is true; hence, by PMI, H_n is true for all positive integers n.	**A1**	
		5	

Q32.

2	We have that $f(1)=9$ is divisible by 9.	**B1**	Checks base case.
	Assume that $f(k)$ is divisible by 9.	**B1**	Makes general statement.
	$f(k+1)+f(k)=2^{3k+3}+8^k+2^{3k}+8^{k-1}=$	**M1**	**Uses** expansion of $f(k+1)$.
	$2^3.2^{3k}+8.8^{k-1}+2^{3k}+8^{k-1}$ OE	**A1**	Correct split of powers
	$=9\left(2^{3k}+8^{k-1}\right)$ OE so $f(k+1)$ is divisible by 9.	**A1**	Alt method: $f(k+1)=2^{3k+3}+8^k$
	So if $f(k)$ is divisible by 9, so is $f(k+1)$, (and $f(1)$ is divisible by 9), $f(n)$ is divisible by 9 for every integer $n\geqslant 1$	**A1**	$=2^3.2^{3k}+8.8^{k-1}$ M1 $=8f(k)$ A1
		6	

7. Proof by induction

Q33.

9(i)	$P_n:\ u_n = 4\left(\frac{5}{4}\right)^n + 3$ Let $n=1$ then $4\left(\frac{5}{4}\right)+3=8 \Rightarrow P_1$ true.	**B1**	States proposition. Proves base case.
	Assume P_k is true for some k. Then	**B1**	States inductive hypothesis.
	$u_{k+1} = \frac{1}{4}\left(5\left(4\left(\frac{5}{4}\right)^k + 3\right) - 3\right) = \textit{correct step}$	**M1**	Proves inductive step.
	$= 4\left(\frac{5}{4}\right)^{k+1} + 3,$	**A1**	
	So $P_k \Rightarrow P_{k+1}$. Therefore, by induction, P_n is true for all positive integers.	**A1**	States conclusion.
9(ii)	$(u_n - 3)x^n = 4x^n\left(\frac{5}{4}\right)^n = 4\left(\frac{5x}{4}\right)^n$ so r $= \left(\frac{5x}{4}\right)$	**M1**	
	So series is convergent for $-1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5}$	**A1**	

9(iii)	$\sum_{n=1}^{N} \ln(u_n - 3) = \sum_{n=1}^{N} \ln\left(4\left(\frac{5}{4}\right)^n\right)$ $= \left(\ln\frac{5}{4}\right)\sum_{n=1}^{N} n + \sum_{n=1}^{N} \ln 4$	**M1**	Alt method: $\sum_{n=1}^{N} \ln(u_n - 3) = ln\prod 4\left(\frac{5}{4}\right)^n$ M1 $= \ln 4^N \prod_1^N \left(\frac{5}{4}\right)^n$ $= N\ln 4 + \ln\left(\frac{5}{4}\right)^{\Sigma n}$
	$= \frac{1}{2}N(N+1)\ln\frac{5}{4} + N\ln 4$ Use $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$.	**M1**	$= N\ln 4 + \frac{N(N+1)}{2}\ln(\frac{5}{4})$ M1
	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln\left(2\sqrt{5}\right) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ oe Alt method: Writes series as an AP M1, uses summation formula M1 Correct answer A1	**A1**	$= N^2 \ln\frac{\sqrt{5}}{2} + N\ln\left(2\sqrt{5}\right) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ A1

Q34.

3(i)	$u_1 < 3 (\text{given})$	**B1**	States base case.
	Assume that $u_k < 3$	**B1**	States inductive hypothesis.
	Then $3 - u_{k+1} = 3 - \frac{4u_k + 9}{u_k + 4} = \frac{-u_k + 3}{u_k + 4} > 0 \Rightarrow u_{k+1} < 3$	**M1 A1**	
	Hence, by induction, $u_n < 3$ for all $n \geqslant 1$.	**B1**	States conclusion.
		5	
3(ii)	$u_{n+1} - u_n = \frac{4u_n + 9}{u_n + 4} - u_n = \frac{-u_n^2 + 9}{u_n + 4}$	**M1 A1**	Considers $u_{n+1} - u_n$.
	So $u_n < 3 \Rightarrow u_{n+1} - u_n > 0$.	**B1**	Uses $u_n < 3$
		3	

7. Proof by induction

Q35.

6	$y^{(1)} = e^x u^{(1)} + ue^x = e^x\left(\binom{1}{0}u + \binom{1}{1}u^{(1)}\right) \Rightarrow H_1$ is true	**M1A1**	Shows base case using product rule
	Assume that $H_k : y^{(k)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right)$	**B1**	States inductive hypothesis.
	Then $y^{(k+1)} = e^x\left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \cdots + \binom{k}{r}u^{(r)} + \cdots + \binom{k}{k}u^{(k)}\right) +$	**M1**	Differentiates using product rule
	$e^x\left(\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \cdots + \binom{k}{r}u^{(r+1)} + \cdots + \binom{k}{k}u^{(k+1)}\right)$ $= e^x\left(\binom{k}{0}u + \ldots\left(\binom{k}{r} + \binom{k}{r-1}\right)u^r + \ldots\binom{k}{k}u^{(k+1)}\right)$	**M1A1**	Shows application of $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.
	$= e^x\left(\binom{k+1}{0}u + \ldots\binom{k+1}{r}u^r + \ldots\binom{k+1}{k+1}u^{(k+1)}\right)$	**B1**	Shows reasoning for first and last term correctly
	So H_k *implies* H_{k+1} so, by induction, H_n is true for all $n \geqslant 1$.	**A1**	States conclusion.
		8	

Q36.

8(i)	$1 = \frac{z^1 - 1}{z - 1}$ So true when $n = 1$.	**B1**	Shows base case.
	Assume that $1 + z + \ldots + z^{k-1} = \frac{z^k - 1}{z - 1}$	**B1**	States inductive hypothesis.
	Then $1 + z + \ldots + z^{k-1} + z^k = \frac{z^k - 1}{z - 1} + z^k = \frac{z^k - 1 + z^k(z-1)}{z-1} = \frac{z^{k+1} - 1}{z - 1}$, so true when $n = k + 1$	**M1 A1**	Combines fractions.
	$H_k \rightarrow H_{k+1}$ Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		5	

8(ii)	Since $\lvert z \rvert < 1$, $\sum_{m=0}^{\infty} z^m = \frac{-1}{z-1}$	**B1**	States $\lvert z \rvert < 1$ and uses formula for sum to infinity of geometric progression.
	$\sum_{m=1}^{\infty} 2^{-m} \sin m\theta = \text{Im}\left(\sum_{m=0}^{\infty} z^m\right) = \text{Im}\left(\frac{-1}{\frac{1}{2}\cos\theta + \text{i}\frac{1}{2}\sin\theta - 1}\right)$	**M1 A1**	Uses de Moivre's theorem.
	$\text{Im}\left(\frac{-\left(\frac{1}{2}\cos\theta - 1 - \text{i}\frac{1}{2}\sin\theta\right)}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}\right)$	**M1**	Multiply numerator and denominator by conjugate.
	$\frac{\frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta} = \frac{2\sin\theta}{5 - 4\cos\theta}$	**A1**	States imaginary part, AG.
		5	

7. Proof by induction

Q37.

1	$3^3-1=26$ is divisible by 13	**B1**	Checks base case.
	Assume that $3^{3k}-1$ is divisible by 13 for some positive integer k	**B1**	States inductive hypothesis.
	Then $3^{3k+3}-1=3^3 3^{3k}-1=26\cdot 3^{3k}+3^{3k}-1$	**M1**	Separates $3^{3k}-1$
	is divisible by 13	**A1**	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n}-1$ is divisible by 13 for every positive integer n.	**A1**	States conclusion.
		5	

Q38.

2	$\frac{dy}{dx}=\frac{a}{ax+1}=(-1)^0\frac{0!a^1}{(ax+1)^1}$ so true for $n=1$.	**M1 A1**	Proves base case.
	Assume that $\frac{d^k y}{dx^k}=(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^k}$ for some positive integer k.	**B1**	States inductive hypothesis.
	Then $\frac{d^{k+1}y}{dx^{k+1}}=-ka(-1)^{k-1}\frac{(k-1)!a^k}{(ax+1)^{k+1}}=(-1)^k\frac{k!a^{k+1}}{(ax+1)^{k+1}}$ so true for $n=k+1$.	**M1 A1**	Differentiates k^{th} derivative.
	By induction, true for every positive integer n.	**A1**	States conclusion.
		6	

Q39.

2(a)	$u_1=1=2^1-1$	**B1**
	Assume that it is true for $n=k$, so $u_k=2^k-1$.	**B1**
	Then $u_{k+1}=2(2^k-1)+1=2^{k+1}-1$	**M1 A1**
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	**A1**
		5
2(b)	$\frac{u_{2n}}{u_n}=\frac{2^{2n}-1}{2^n-1}=\frac{(2^n-1)(2^n+1)}{(2^n-1)}=2^n+1$	**M1 A1**
		2

7. Proof by induction

Q40.

5	$\frac{d}{dx}(x\sin x)=x\cos x+\sin x=(-1)^0(x\cos x+(2(1)-1)\sin x)$	**B1**	Checks base case using product rule.
	Assume true for $n=k$, so $\frac{d^{2k-1}}{dx^{2k-1}}(x\sin x)=(-1)^{k-1}(x\cos x+(2k-1)\sin x)$	**B1**	States inductive hypothesis.
	Then $\frac{d^{2k}}{dx^{2k}}(x\sin x)=(-1)^{k-1}(-x\sin x+2k\cos x)$	**M1 A1**	Differentiates once. Must have correct LHS for A1.
	$\frac{d^{2k+1}}{dx^{2k+1}}(x\sin x)=(-1)^{k-1}(-x\cos x-\sin x-2k\sin x)$ $=(-1)^k(x\cos x+(2k+1)\sin x)$	**M1 A1**	Differentiates again.
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		7	

Q41.

2	$7^2-1=48$ is divisible by 12.	**B1**	Checks base case.
	Assume that $7^{2k}-1$ is divisible by 12 for **some** positive integer k.	**B1**	States inductive hypothesis.
	Then $7^{2k+2}-1=49\cdot7^{2k}-1=48\cdot7^{2k}+7^{2k}-1$ Or $(7^{2k+2}-1)-(7^{2k}-1)=48\cdot7^{2k}$	**M1**	Separates $7^{2k}-1$.
	$7^{2k+2}-1$ is divisible by 12.	**A1**	
	So if is true for $n=k$ (could be written earlier) it is also true for $n=k+1$. Hence, by induction, true for every positive integer n.	**A1**	Depends on previous M1 and A1.
		5	

Q42.

1	$2^4+31-2=45$ is divisible by 15	**B1**	Checks base case.
	Assume that $2^{4k}+31^k-2$ is divisible by 15 for some positive integer k.	**B1**	States inductive hypothesis.
	Then $2^{4k+4}+31^{k+1}-2=(15+1)2^{4k}+(30+1)31^k-2$	**M1 A1**	Separates $2^{4k}+31^k-2$ or considers difference.
	is divisible by 15 because $15\times2^{4k}+30\times31^k$ is divisible by 15.	**A1**	
	Hence, by induction, true for every positive integer n.	**A1**	
		6	

7. Proof by induction

Q43.

3(a)	$5\times1^4+1^2=\frac{1}{2}(2)^2(2+1)[=6]$ so H_1 is true.	**B1**	Checks base case.
	Assume that $\sum_{r=1}^{k}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)$	**B1**	States inductive hypothesis [for some k] including the algebraic form. If says for **ALL** k, then B0.
	$\sum_{r=1}^{k+1}\left[\left(5r^4+r^2\right)\right]=\frac{1}{2}k^2(k+1)^2(2k+1)+5(k+1)^4+(k+1)^2$	**M1**	Considers sum to $k+1$.
	$\frac{1}{2}(k+1)^2\left(2k^3+k^2+10(k+1)^2+2\right)$	**M1**	Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.
	$\frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12)=\frac{1}{2}(k+1)^2(k+2)^2(2k+3)$	**A1**	Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.
	So H_{k+1} is true. By induction, H_n is true for all positive integers n.	**A1**	States conclusion. Implication must be clearly expressed.
		6	
3(b)	$5\sum_{r=1}^{n}r^4+\frac{1}{6}n(n+1)(2n+1)=\frac{1}{2}n^2(n+1)^2(2n+1)$	**M1**	Uses correct formula for $\sum r^2$.
	$[5]\sum_{r=1}^{n}r^4=\frac{1}{6}n(n+1)(2n+1)(3n(n+1)-1)$	**M1**	Makes $\sum r^4$ the subject and takes out all linear factors and the remaining term is of correct form.
	$\sum_{r=1}^{n}r^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$	**A1**	CAO
		3	

Q44.

3(a)	$a_2=2^3$ leading to $\ln a_2=3\ln 2$ so true when $n=2$.	**B1**	Checks base case. Must be n =2 and exact.
	Assume that $\ln a_k \geqslant 3^{k-1}\ln 2$.	**B1**	States inductive hypothesis.
	$\ln a_{k+1}=3\ln\left(a_k+\frac{1}{a_k}\right)>3\ln a_k$	**M1**	Applies result given in part **(a)**.
	$\geqslant 3^k\ln 2$	**M1 A1**	Applies inductive hypothesis and writes in required form.
	So true when $n=k+1$. By induction, true for all integers $n \geqslant 2$.	**A1**	States conclusion, all previous marks must be gained.
		6	
3(b)	$3\ln\left(a_n+\frac{1}{a_n}\right)-\ln a_n>2\ln a_n$	**M1**	Applies results given in parts **(a)** and **(b)**.
	$>2\times3^{n-1}\ln 2=3^{n-1}\ln 4$	**A1**	AG
		2	

7. Proof by induction

Q45.

2	$\frac{dy}{dx} = axe^{ax} + e^{ax} = (ax+1)e^{ax}$ so true when $n=1$.	**M1 A1**	Differentiates once using the product rule.
	Assume that $\frac{d^k y}{dx^k} = \left(a^k x + ka^{k-1}\right)e^{ax}$.	**B1**	States inductive hypothesis.
	$\frac{d^{k+1} y}{dx^{k+1}} = a\left(a^k x + ka^{k-1}\right)e^{ax} + e^{ax}\left(a^k\right) = \left(a^{k+1} + (k+1)a^k\right)e^{ax}$	**M1 A1**	Differentiates kth derivative.
	So true when $n = k+1$. By induction, true for all positive integers n.	**A1**	States conclusion.
		6	

Q46.

3(a)	$u_1 > 4$ (given)	**B1**	States base case.
	Assume that $u_k > 4$ for some positive integer k.	**B1**	States inductive hypothesis.
	Then $u_{k+1} - 4 = \frac{u_k^2 + u_k + 12}{2u_k} - 4 = \frac{u_k^2 - 7u_k + 12}{2u_k}$	**M1**	Considers $u_{k+1} - 4$, puts over common denominator.
	$u_{k+1} - 4 = \frac{(u_k - 3)(u_k - 4)}{2u_k} > 0$	**A1**	
	Hence, by induction, $u_n > 4$ for all positive integers n.	**A1**	
		5	
3(b)	$u_{n+1} - u_n = \frac{-u_n^2 + u_n + 12}{2u_n} = \frac{-(u_n - 4)(u_n + 3)}{2u_n}$	**M1 A1**	Considers $u_{n+1} - u_n$.
	So $u_n > 4 \Rightarrow u_{n+1} - u_n < 0$	**A1**	Uses $u_n > 4$.
		3	

Q47.

5(b)	$\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ so true when $n=1$.	**B1**	States base case.
	Assume that it is true for $n=k$, so $\mathbf{A}^k = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}$.	**B1**	States inductive hypothesis.
	Then $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+ak \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (k+1)a \\ 0 & 1 \end{pmatrix}$	**M1 A1**	Multiplies $\mathbf{A}^k$ with $\mathbf{A}$.
	If it is true for [$n=1$ and] $n=k$ then it is also true for $n=k+1$. Hence, [by induction,] true for all positive integers.	**A1**	Everything correct and states conclusion.
		5	

7. Proof by induction

Q48.

2	$7^2+97-50=96$ is divisible by 48	**B1**	Checks base case.
	Assume that $7^{2k}+97^k-50$ is divisible by 48 for some positive integer k.	**B1**	States inductive hypothesis.
	Then $7^{2k+2}+97^{k+1}-50=(48+1)7^{2k}+(96+1)97^k-50$	**M1 A1**	Separates $7^{2k}+97^k-50$ or considers difference.
	is divisible by 48 because $48\times7^{2k}+96\times97^k$ is divisible by 48.	**A1**	
	Hence, by induction, true for every positive integer n.	**A1**	
		6	

Q49.

4	$\frac{d}{dx}(xf(x))=xf'(x)+f(x)=xf'(x)+(2(1)-1)f(x)$	**B1**	Checks base case using product rule.
	Assume true for $n=k$, so $\frac{d^{2k-1}}{dx^{2k-1}}(xf(x))=xf'(x)+(2k-1)f(x)$	**B1**	States inductive hypothesis.
	Then $\frac{d^{2k}}{dx^{2k}}(xf(x))=xf(x)+2kf'(x)$	**M1 A1**	Differentiates once.
	$\frac{d^{2k+1}}{dx^{2k+1}}(xf(x))=xf'(x)+f(x)+2kf(x)$ $=xf'(x)+(2k+1)f(x)$	**M1 A1**	Differentiates again.
	So, it is also true for $n=k+1$. Hence, by induction, true for all positive integers.	**A1**	States conclusion.
		7	